

On the Mathematical Structure of the Fundamental Forces of Nature

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The main idea of this article is based on my previous articles (references [1], [2], [3]). In this work by introducing a new mathematical approach based on the algebraic structure of integers (the domain of integers), and assuming the “discreteness” of physical quantities such as the components of the relativistic n -momentum, we derive all the mathematical laws governing the fundamental forces of nature. These obtained laws that are unique, distinct and in the form of the complex tensor equations, represent the force of gravity, the electromagnetic force, and the (strong) nuclear force (and only these three kinds of forces, for all dimensions $D \geq 2$). Each tensor equation contains the term of the mass m_0 (as the invariant mass of the supposed force carrier particle), as well as the term of the external current (as the external source of the force field). In some special cases, these tensor equations are turned into the wave equations that are similar to the Pauli and Dirac equations. In fact, the mathematical laws obtained in this paper, are the corrected and generalized forms of the current field equations including Maxwell equations, Yang-Mills equations and Einstein equations, as well as (in some special conditions) Pauli equation, Dirac equation, and so on. A direct proof of the absence of magnetic monopoles in nature is one of the outcomes of this research, according to the unique formulations of the laws of the fundamental forces that we have derived.

1. Introduction

One of the biggest ontological questions might be that: “Why the universe and the fundamental forces that are acting in it, are in the way and form, which we realize them?”; the forces that are the causers and the governors of all of the movements, the actions and reactions in the physical world. In this article, we are going to consider this question by a logical approach.

This paper is based on my previous articles ([1], [2], [3]), as well as my thesis work, (1997) [4], (but in a new expanded framework). Here by representing a new axiomatic mathematical approach, as well as assuming the discreteness of the components of the relativistic n -momentum, the basic laws governing the fundamental forces of nature will be derived. These laws are unique and in the form of tensor equations. The main results of this paper include:

1-1. The mathematical laws governing the fundamental forces of nature (for all dimensions):

$$D_{[\lambda} R_{\mu\nu]\rho\sigma} = 0, \quad (1-1)$$

$$D_{\mu}^* R^{\mu}_{\nu\rho\sigma} = -J_{\nu\rho\sigma}^{(G)} \quad (1-2)$$

$$D_{[\lambda} Z_{\mu\nu]\rho} = 0, \quad (2-1)$$

$$D_{\mu}^* Z^{\mu}_{\nu\rho} = -J_{\nu\rho}^{(N)} \quad (2-2)$$

$$D_{[\lambda} F_{\mu\nu]} = 0, \quad (3-1)$$

$$D_{\mu}^* F^{\mu}_{\nu} = -J_{\nu}^{(E)} \quad (3-2)$$

where $F_{\mu\nu}$ is the field tensor of the electromagnetic force, $Z_{\mu\nu\rho}$ is the field tensor of the nuclear (strong) force, and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the gravitational field, and

$$D_{\mu} = \nabla_{\mu} + \frac{im_0}{\hbar} k_{\mu}, \quad D_{\mu}^* = \nabla_{\mu} - \frac{im_0}{\hbar} k_{\mu} \quad (4)$$

$$\mu = 0: \quad k_{\mu} = \frac{1}{\sqrt{g^{00}}}, \quad (5)$$

$$\mu \neq 0: \quad k_{\mu} = 0,$$

where m_0 is the invariant mass of the (supposed) force carrier particle.

As we will show in the section 3., equations (1-1) – (1-2) (representing the gravitational fields), also could be written as follows,

for two dimensional case ($D = 2$) we have

$$R - \Lambda = -8\pi T \quad (6)$$

$$R_{\mu\nu} = -\frac{1}{2}(8\pi T)g_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu} \quad (7)$$

$$8\pi T_{\mu\nu} + \frac{im_0}{\hbar} K_{\mu\nu} = \frac{1}{2}(8\pi T)g_{\mu\nu} \quad (8)$$

and for higher dimensions ($D > 2$) we have

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi T_{\mu\nu} - \frac{im_0}{\hbar} K_{\mu\nu} - \Lambda g_{\mu\nu} \quad (9)$$

where $K_{\mu\nu} = \nabla_\mu k_\nu$.

1-2. According to the unique structure of the tensor equations (3-1) and (3-2) that in fact, are the corrected and generalized form of the Maxwell equations, we will conclude that there are not any magnetic monopole in nature.

In the next section, we describe the bases of the linearization theory, and in the section 3. we show its applications in physics.

2. Theory of Linearization in the Domain of Integers: As a New Axiomatic Mathematical Approach

The algebraic axioms of the domain of integers Z , with binary operations $(+, \times)$, usually are defined as follows [5]:

- $a_1, a_2, a_3, \dots \in Z$,

- Closer: $a_k + a_l \in Z, \quad a_k \times a_l \in Z \quad (10)$

- Associativity: $a_k + (a_l + a_r) = (a_k + a_l) + a_r, \quad a_k \times (a_l \times a_r) = (a_k \times a_l) \times a_r \quad (11)$

- Commutativity: $a_k + a_l = a_l + a_k, \quad a_k \times a_l = a_l \times a_k \quad (12)$

- Existence of an identity element: $a_k + 0 = a_1, \quad a_k \times 1 = a_k \quad (13)$

$$\text{- Existence of inverse element (for addition): } a_k + (-a_k) = 0 \quad (14)$$

$$\begin{aligned} \text{- Distributivity: } \quad a_k \times (a_l + a_r) &= (a_k \times a_l) + (a_k \times a_r), \\ (a_k + a_l) \times a_r &= (a_k \times a_r) + (a_l \times a_r) \end{aligned} \quad (15)$$

$$\text{- No zero divisors: } (a_k = 0 \vee a_l = 0) \Leftrightarrow a_k \times a_l = 0, \quad (16-1)$$

equivalently, the axiom (16-1) could be defined as:

$$\begin{aligned} &[(a_1 \times m_1 = 0, m_1 \neq 0) \vee (a_2 \times m_2 = 0, m_2 \neq 0) \vee \dots \vee \\ &\vee (a_r \times m_r = 0, m_r \neq 0)] \Leftrightarrow a_1 \times a_2 \times a_3 \times \dots \times a_r = 0. \end{aligned} \quad (16-2)$$

If we just suppose $[a]_{1 \times 1} (\equiv a), [b]_{1 \times 1} (\equiv b), [c]_{1 \times 1} (\equiv c), \dots \in Z_{1 \times 1} (\equiv Z)$, then equivalently, the axioms (10) – (15) could also be written by quadratic matrices (with integer components) as follows:

$$\text{- } M_k = [m_{kij}] , \quad m_{kij} \in Z, \quad \exists n \in N : i, j = 1, 2, 3, \dots, n, \quad M_1, M_2, M_3, \dots \in Z_{n \times n},$$

$$\text{- Closer: } M_k + M_l \in Z_{n \times n}, \quad M_k \times M_l \in Z_{n \times n} \quad (17)$$

$$\text{- Associativity: } M_k + (M_l + M_r) = (M_k + M_l) + M_r, \quad M_k \times (M_l \times M_r) = (M_k \times M_l) \times M_r \quad (18)$$

$$\text{- Commutativity (for addition): } M_k + M_l = M_l + M_k \quad (19-1)$$

- Property of the transpose for matrix multiplication:

$$(M_k \times M_l)^T = M_l^T \times M_k^T \quad (19-2)$$

where M_k^T is the transpose of matrix M_k .

$$\text{- Existence of an identity element: } M_k + 0 = M_k, \quad M_k \times I_{n \times n} = M_k \quad (20)$$

$$\text{- Existence of the inverse element (for addition): } M_k + (-M_k) = 0 \quad (21)$$

$$\begin{aligned} \text{- Distributivity: } \quad M_k \times (M_l + M_r) &= (M_k \times M_l) + (M_k \times M_r), \\ (M_k + M_l) \times M_r &= (M_k \times M_r) + (M_l \times M_r); \end{aligned} \quad (22)$$

From the axioms (10) – (15), we can obtain the axioms (17) – (22) and vice versa.

In this article, we introduce the following algebraic axiom as a new property of integers, and we add it to the axioms (17) – (22) (this new axiom is somehow the generalized form of the axiom (16-2), and in fact, the axiom (16-2) will be replaced with the axiom (23)):

““ If we assume the algebraic form $F(b_{pq}) = \sum_{q=1}^s \prod_{p=1}^r b_{pq}$ and the $n \times n$ quadratic matrices $A_k = [a_{k_{ij}}]$,

where $H_{k_{ij}pq}$ are some coefficients, and

$$a_{k_{ij}} = \sum_{q=1}^s \sum_{p=1}^r H_{k_{ij}pq} b_{pq}, \quad b_{pq}, H_{k_{ij}pq} \in Z \ (\equiv Z_{1 \times 1}), \quad i, j = 1, 2, 3, \dots, n, \quad k = 1, 2, 3, \dots, r,$$

$$p = 1, 2, 3, \dots, r, \quad q = 1, 2, 3, \dots, s,$$

then we have the following axiom:

$$\begin{aligned} \exists M_k \in Z_{n \times n}, \quad & [(A_1 \times M_1 = 0, M_1 \neq 0) \vee (A_2 \times M_2 = 0, M_2 \neq 0) \vee \dots \vee (A_r \times M_r = 0, M_r \neq 0)] \wedge \\ & \wedge (A_1 \times A_2 \times A_3 \times \dots \times A_r = F(b_{pq}) I_{n \times n})] \Leftrightarrow F(b_{pq}) = 0. \end{aligned} \quad (23)$$

Remark 1. In (23), according to the arbitrariness of all of the components of the $n \times n$ matrix M_k , without loss of generality, we may replace the $n \times n$ matrix M_k with a $n \times 1$ matrix M_k , in each of the equations $A_k M_k = 0$ (with the same condition $M_k \neq 0$, but only with the “ n ” number of arbitrary components).

Note that the integer elements $a_{k_{ij}}$ are the “linear” forms of the integer elements b_{pq} .

We can obtain the axiom (16-1) (or its equivalent, the axiom (16-2)) from the axiom (23), **but not vice versa**. Only for the special case $n = 1$, the set of axioms (17) – (23) becomes equivalent to the set of axioms (10) – (16-2). Definitely, the axiom (23) is a new axiom and in this section and the next section we’ll demonstrate some of its outcomes and applications.

Generally, there are standard and specific methods, approaches and procedures for considering and solving the linear equations in the set of integers [7]. Since the necessary and sufficient condition for an equation of the r^{th} order such as $F(b_{pq}) = 0$ (in the domain of integers) is the transforming (or converting or in fact, “**linearizing**”) it into a system of linear equations of the type $A_k M_k = 0$ (according to the axiom (23), where $M_k \neq 0$, $M_k : n \times 1$ matrix), naturally, the main application of the axiom (23) will be the transforming the higher order equations into the corresponding systems of linear equations. In this section, based on (23), essentially, we’ll obtain the systems of linear equations that correspond to the second order equation of the form: $F(b_{pq}) = 0$, and also some of the higher order equations. In the methodological point of view, firstly, for obtaining and specifying a system of linear equations that corresponds to a given equation of the type $F(b_{pq}) = 0$ (defined in (23)), we assume and consider the minimum value for n (the size number of $n \times n$ matrices A_k). Secondly, by replacing the

components of the matrices A_k with the linear forms $a_{k_{ij}} = \sum_{q=1}^s \sum_{p=1}^r H_{k_{ij}pq} b_{pq}$, we calculate the product

$\prod_{k=1}^r A_k$, and then we put it equal to the matrix $F(b_{pq})I_{n \times n}$. Then using this (obtained) equation, we can calculate the coefficients $H_{k_{ij}pq}$ (which are independent of elements b_{pq}). Through this, easily, the coefficients $H_{k_{ij}pq}$ are calculated and obtained by routine and standard methods of solving the equations (in the set of integers). Thirdly, the algebraic forms

$$F(b_{pq}) = \sum_{q=1}^s \prod_{p=1}^r b_{pq} \quad (24)$$

via some certain rules and linear transformations, could be transformed into the algebraic forms of the type

$$G(c_1, c_2, c_3, \dots, c_s) = \sum_{i_1, i_2, i_3, \dots, i_r=1}^s B_{i_1 i_2 i_3 \dots i_r} \prod_{p=1}^r c_{i_p} \quad (25)$$

In continuation, by some examples we will show how the forms $F(b_{pq}) = \sum_{q=1}^s \prod_{p=1}^r b_{pq}$ could be transformed into the forms $G(c_1, c_2, c_3, \dots, c_s)$ (through some linear transformations). Furthermore, as we'll show that, exceptionally, the second order forms of (24) could be transformed into the following quadratic forms (by similar linear transformation)

$$G(c_1, c_2, c_3, \dots, c_s, d_1, d_2, d_3, \dots, d_s) = \sum_{i_1, i_2=1}^s B_{i_1 i_2} \prod_{p=1}^2 c_{i_p} - \sum_{i_1, i_2=1}^s B_{i_1 i_2} \prod_{p=1}^2 d_{i_p} \quad (26)$$

Remark 2. Concerning the formula (24), it is easy to show that

$$\left(\sum_{q=1}^s \prod_{p=1}^r b_{pq} c_{(r+1)q} = 0, \sum_{q=1}^s \prod_{p=1}^r b_{pq} d_{(r+1)q} = 0 \right) \Rightarrow \sum_{q=1}^s \prod_{p=1}^r b_{pq} (c_{(r+1)q} + d_{(r+1)q}) = 0, \quad (24-1)$$

and

$$\sum_{q=1}^s \prod_{p=1}^r b_{pq} c_{(r+1)q} = 0 \Leftrightarrow \sum_{q=1}^s \prod_{p=1}^r b_{pq} (t c_{(r+1)q}) = 0. \quad (24-2)$$

where the parameter t is an arbitrary integer, with condition $t \neq 0$. Below we write the systems of linear equations that correspond with some special cases of the following equation (that according to the axiom (23), one system for each case is enough):

$$F(b_{pq}) = \sum_{q=1}^s \prod_{p=1}^r b_{pq} = 0 \quad (24-3)$$

that has been indicated in (23). Some special cases of (24-3), which we will consider below, in particular include the second order equations with the different number of the elements, as well as some of the other higher order equations.

For $s = 1, 2, 3, \dots$, $r = 2$, equation (24-3) becomes as follows, respectively,

$$\sum_{q=1}^1 \prod_{p=1}^2 b_{pq} = b_{11}b_{21} = 0, \quad (27)$$

$$\sum_{q=1}^2 \prod_{p=1}^2 b_{pq} = b_{11}b_{21} + b_{12}b_{22} = 0, \quad (28)$$

$$\sum_{q=1}^3 \prod_{p=1}^2 b_{pq} = b_{11}b_{21} + b_{12}b_{22} + b_{13}b_{23} = 0, \quad (29)$$

$$\sum_{q=1}^4 \prod_{p=1}^2 b_{pq} = b_{11}b_{21} + b_{12}b_{22} + b_{13}b_{23} + b_{14}b_{24} = 0, \quad (30)$$

$$\sum_{q=1}^5 \prod_{p=1}^2 b_{pq} = b_{11}b_{21} + b_{12}b_{22} + b_{13}b_{23} + b_{14}b_{24} + b_{15}b_{25} = 0; \quad (31)$$

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Now, a matrix equation (here we mean a system of linear equations) corresponding to (27) (according to (23)) is

$$\begin{bmatrix} e_0 & 0 \\ 0 & f_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = 0 \quad (32)$$

where $e_0 = b_{11}$, $f_0 = b_{21}$;

Similarly, for (28) we have the following matrix equation

$$\begin{bmatrix} 0 & 0 & e_0 & f_1 \\ 0 & 0 & -e_1 & f_0 \\ f_0 & f_1 & 0 & 0 \\ -e_1 & e_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = 0 \quad (33)$$

where $e_0 = b_{11}, f_0 = b_{21}, e_1 = b_{12}, f_1 = b_{22}$;

Using (33) we may get equivalently, the following matrix equation for (28)

$$\begin{bmatrix} e_0 & f_1 \\ -e_1 & f_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = 0 \quad (34)$$

where $e_0 = b_{11}, f_0 = b_{21}, e_1 = b_{12}, f_1 = b_{22}$;

A system of linear equations corresponding to (29) is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & e_0 & 0 & -e_2 & f_1 \\ 0 & 0 & 0 & 0 & 0 & e_0 & -e_1 & -f_2 \\ 0 & 0 & 0 & 0 & f_2 & f_1 & f_0 & 0 \\ 0 & 0 & 0 & 0 & -e_1 & e_2 & 0 & f_0 \\ -f_0 & 0 & -f_2 & -e_1 & 0 & 0 & 0 & 0 \\ 0 & -f_0 & f_1 & -e_2 & 0 & 0 & 0 & 0 \\ e_2 & -e_1 & -e_0 & 0 & 0 & 0 & 0 & 0 \\ f_1 & f_2 & 0 & -e_0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{bmatrix} = 0 \quad (35)$$

where $e_0 = b_{11}, f_0 = b_{21}, e_1 = b_{12}, f_1 = b_{22}, e_2 = b_{13}, f_2 = b_{23}$;

from (35) we can obtain the following matrix equation for equation (29)

$$\begin{bmatrix} e_0 & 0 & -e_2 & f_1 \\ 0 & e_0 & -e_1 & -f_2 \\ f_2 & f_1 & f_0 & 0 \\ -e_1 & e_2 & 0 & f_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = 0 \quad (36)$$

where $e_0 = b_{11}, f_0 = b_{21}, e_1 = b_{12}, f_1 = b_{22}, e_2 = b_{13}, f_2 = b_{23}$;

Similarly, how we obtained the matrix equations (35) and (36) for equations (28) and (29), the matrix equations corresponding to (30) and (31) are obtained as follows,

for (30) we get

$$\begin{bmatrix} e_0 & 0 & 0 & 0 & 0 & -e_3 & e_2 & f_1 \\ 0 & e_0 & 0 & 0 & e_3 & 0 & -e_1 & f_2 \\ 0 & 0 & e_0 & 0 & -e_2 & e_1 & 0 & f_3 \\ 0 & 0 & 0 & e_0 & -f_1 & -f_2 & -f_3 & 0 \\ 0 & -f_3 & f_2 & e_1 & f_0 & 0 & 0 & 0 \\ f_3 & 0 & -f_1 & e_2 & 0 & f_0 & 0 & 0 \\ -f_2 & f_1 & 0 & e_3 & 0 & 0 & f_0 & 0 \\ -e_1 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & f_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{bmatrix} = 0 \quad (37)$$

where $e_0 = b_{11}, f_0 = b_{21}, e_1 = b_{12}, f_1 = b_{22}, e_2 = b_{13}, f_2 = b_{23}, e_3 = b_{14}, f_3 = b_{24}$;

and for (31) we obtain

$$\begin{bmatrix} e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_4 & 0 & e_3 & -e_2 & f_1 \\ 0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & -e_3 & 0 & -e_1 & -f_2 \\ 0 & 0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & -e_2 & -e_1 & 0 & f_3 \\ 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & 0 & f_1 & -f_2 & -f_3 & 0 \\ 0 & 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_3 & -e_2 & -e_1 & 0 & 0 & 0 & -f_4 \\ 0 & 0 & 0 & 0 & 0 & e_0 & 0 & 0 & e_3 & 0 & f_1 & -f_2 & 0 & 0 & f_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0 & 0 & -e_2 & -f_1 & 0 & -f_3 & 0 & -f_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & -e_1 & f_2 & f_3 & 0 & f_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -f_4 & 0 & -f_3 & f_2 & f_1 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f_4 & 0 & -f_3 & 0 & e_1 & -e_2 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -f_4 & 0 & 0 & f_2 & -e_1 & 0 & -e_3 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 \\ f_4 & 0 & 0 & 0 & f_1 & e_2 & e_3 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 \\ 0 & f_3 & f_2 & -e_1 & 0 & 0 & 0 & -e_4 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 \\ -f_3 & 0 & f_1 & e_2 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 \\ f_2 & f_1 & 0 & e_3 & 0 & -e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 \\ -e_1 & e_2 & -e_3 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{15} \\ m_{16} \end{bmatrix} = 0 \quad (38)$$

where $e_0 = b_{11}, f_0 = b_{21}, e_1 = b_{12}, f_1 = b_{22}, e_2 = b_{13}, f_2 = b_{23}, e_3 = b_{14}, f_3 = b_{24}, e_4 = b_{15}, f_4 = b_{25}$;

Similarly, systems of linear equations with larger sizes could be obtained for the equation (24-3) (where $s = 1, 2, 3, \dots$, $r = 2$).

In general, the size of the quadratic matrices of these matrix equations (corresponding with the second order equations of the type (24-3), i.e. for $r = 2$) is $2^s \times 2^s$. But exceptionally, this size is reducible to $2^{s-1} \times 2^{s-1}$ (only for the case of the second order equations), as it was for equations (29) – (31).

Generally, the size of the quadratic matrices of the matrix equations (corresponding with the general cases of the equation (24-3)) is $r^s \times r^s$. For all values of s , r , these matrix equations (according to (23), and corresponding to the general cases of the equation (24-3)) are derivable and calculable.

Meanwhile, as we previously noted, any of quadratic matrices (i.e. the matrices A_k in (23)) in equations (32) – (38) and so on, is just one of the possible matrices, from which we may obtain. Definitely, there are other matrices (different ones, but with the same size and similar structure) which we may obtain for constructing other equivalent cases (based on (23)) of the equations (31) – (38); we just selected those individual matrices (in (32) – (38)) because of their particular structures, as well as their particular applications that will be in the next section.

As some examples of the third order cases of equation (24-3), the systems of linear equations corresponding with two equations

$$\sum_{q=1}^1 \prod_{p=1}^3 b_{pq} = b_{11}b_{21}b_{31}, \quad (39)$$

$$\sum_{q=1}^2 \prod_{p=1}^3 b_{pq} = b_{11}b_{21}b_{31} + b_{12}b_{22}b_{32}; \quad (40)$$

respectively, are

$$\begin{bmatrix} e_0 & 0 & 0 \\ 0 & f_0 & 0 \\ 0 & 0 & g_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0, \quad (41)$$

$$\begin{bmatrix} 0 & 0 & 0 & e_1 & 0 & 0 & 0 & 0 & e_0 \\ 0 & 0 & 0 & 0 & f_1 & 0 & f_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_1 & 0 & g_0 & 0 \\ 0 & 0 & e_0 & 0 & 0 & 0 & e_1 & 0 & 0 \\ f_0 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 & 0 \\ 0 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 \\ e_1 & 0 & 0 & 0 & 0 & e_0 & 0 & 0 & 0 \\ 0 & f_1 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_1 & 0 & g_0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \end{bmatrix} = 0 \quad (42)$$

where $e_0 = b_{11}, f_0 = b_{21}, g_0 = b_{31}, e_1 = b_{12}, f_1 = b_{22}, g_1 = b_{32}$.

The size of the quadratic matrix in the matrix equation, corresponding to the next 3rd order equation, i.e.

$$\sum_{q=1}^3 \prod_{p=1}^3 b_{pq} = 0, \text{ is } 27 \times 27.$$

Concerning the fourth order cases of equation (24-3), such as

$$\sum_{q=1}^1 \prod_{p=1}^4 b_{pq} = b_{11} b_{21} b_{31} b_{41}, \quad (43)$$

$$\sum_{q=1}^2 \prod_{p=1}^4 b_{pq} = b_{11} b_{21} b_{31} b_{41} + b_{12} b_{22} b_{32} b_{42}; \quad (44)$$

respectively, the systems of linear equations corresponding to them are

$$\begin{bmatrix} e_0 & 0 & 0 & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & g_0 & 0 \\ 0 & 0 & 0 & h_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = 0, \quad (45)$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & -e_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 \\
0 & 0 & 0 & 0 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_1 & 0 & 0 & 0 & 0 & 0 & 0 & g_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & h_0 & 0 \\
0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & -e_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & -e_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 \\
-e_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 \\
0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_1 & 0 & 0 & 0 & 0 & 0 & 0 & g_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & h_0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{15} \\ m_{16} \end{bmatrix} = 0 \quad (46)$$

where $e_0 = b_{11}, f_0 = b_{21}, g_0 = b_{31}, h_0 = b_{41}, e_1 = b_{12}, f_1 = b_{22}, g_1 = b_{32}, h_1 = b_{42}$.

Similarly, as we noted above, for the fifth and the higher order cases of (24-3), with the larger number of unknown elements, we get the matrix equations including the quadratic matrices with the size $r^s \times r^s$.

Meanwhile, we can use the following linear relations (as the general rules) for transforming the second, the third, the forth and the higher order cases of the form (24) into the form (25); e.g. for the second order

$$\sum_{i_1, i_2=1}^s B_{i_1 i_2} \prod_{p=1}^2 c_{i_p} = \sum_{q=1}^s \prod_{p=1}^2 b_{pq}, \quad (47)$$

we have

$$b_{11} = c_1, \quad b_{21} = \sum_{i_2=1}^s B_{1i_2} c_{i_2}, \quad b_{12} = c_2, \quad b_{22} = \sum_{i_2=1}^s B_{2i_2} c_{i_2}, \quad \dots, \quad b_{1s} = c_s, \quad b_{2s} = \sum_{i_2=1}^s B_{si_2} c_{i_2}; \quad (48)$$

and for the third order:

$$\sum_{i_1, i_2, i_3=1}^s B_{i_1 i_2 i_3} \prod_{p=1}^3 c_{i_p} = \sum_{q=1}^s \prod_{p=1}^3 b_{pq}, \quad (49)$$

we have

$$\begin{aligned} b_{11} = c_1, \quad b_{21} = c_1, \quad b_{31} = \sum_{i_3=1}^s B_{11i_3} c_{i_3}, \quad b_{12} = c_1, \quad b_{22} = c_2, \quad b_{23} = \sum_{i_3=1}^s B_{12i_3} c_{i_3}, \\ ,..., \quad b_{1s} = c_1, \quad b_{2s} = c_s, \quad b_{3s} = \sum_{i_3=1}^s B_{1si_3} c_{i_3}, ..., \quad b_{1(s^2-s+1)} = c_s, \quad b_{2(s^2-s+1)} = c_1, \quad b_{3(s^2-s+1)} = \sum_{i_3=1}^s B_{s1i_3} c_{i_3}, \\ ,..., \quad b_{1(s^2)} = c_s, \quad b_{2(s^2)} = c_s, \quad b_{3(s^2)} = \sum_{i_3=1}^s B_{ss i_3} c_{i_3}. \end{aligned} \quad (50)$$

Similarly, for transforming the forth order and the higher order cases of equation (24) into (25), we can define some linear transformations such as (48) and (50). However, if necessary, it is possible for transforming (24) into (25), we define other linear transformations, as well.

According to the particular applications of the second order cases of the equation (24-3) in the next section (the section 3.), here we do consider and introduce some of the properties of the matrix equations (34), (36), (37), (38).

First, we consider the following equation (which its left part is a special case of (26)):

$$\sum_{i,j=0}^n B_{ij} (c_i c_j - d_i d_j) = 0 \quad (51)$$

Where $B_{ij} = B_{ji}$. Now using the matrix equations (34) (for $n=1$), (36) (for $n=2$), (37) (for $n=3$) and

(38) (for $n=4$) and so on, as well as the linear transformations of the type $e_i = \sum_{j=0}^n B_{ij} (c_j + d_j)$,

$f_i = c_i - d_i$, or

$$\begin{bmatrix} f_0 \\ f_1 \\ f_3 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix} = \begin{bmatrix} c_0 - d_0 \\ c_1 - d_1 \\ c_2 - d_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n - d_n \end{bmatrix}, \quad (52-1)$$

$$\begin{bmatrix} e_0 \\ e_1 \\ e_3 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} & B_{02} & \cdot & \cdot & \cdot & B_{0n} \\ B_{10} & B_{11} & B_{12} & \cdot & \cdot & \cdot & B_{1n} \\ B_{20} & B_{21} & B_{22} & \cdot & \cdot & \cdot & B_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_{n0} & B_{n1} & B_{n2} & \cdot & \cdot & \cdot & B_{nn} \end{bmatrix} \begin{bmatrix} c_0 + d_0 \\ c_1 + d_1 \\ c_2 + d_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n + d_n \end{bmatrix}; \quad (52-2)$$

we obtain the following matrix equations that correspond to equation (51), respectively

$$[B_{00}(c_0 + d_0)]m_1 = 0, \quad (53)$$

$$\begin{bmatrix} \sum_{j=0}^1 B_{0j}(c_j + d_j) & c_1 - d_1 \\ -\sum_{j=0}^1 B_{1j}(c_j + d_j) & c_0 - d_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = 0, \quad (54)$$

$$\begin{bmatrix} \sum_{j=0}^2 B_{0j}(c_j + d_j) & 0 & -\sum_{j=0}^2 B_{2j}(c_j + d_j) & c_1 - d_1 \\ 0 & \sum_{j=0}^2 B_{0j}(c_j + d_j) & -\sum_{j=0}^2 B_{1j}(c_j + d_j) & -(c_2 - d_2) \\ c_2 - d_2 & c_1 - d_1 & c_0 - d_0 & 0 \\ -\sum_{j=0}^2 B_{1j}(c_j + d_j) & \sum_{j=0}^2 B_{2j}(c_j + d_j) & 0 & c_0 - d_0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = 0, \quad (55)$$

$$\begin{bmatrix}
e_0 & 0 & 0 & 0 & 0 & -e_3 & e_2 & f_1 \\
0 & e_0 & 0 & 0 & e_3 & 0 & -e_1 & f_2 \\
0 & 0 & e_0 & 0 & -e_2 & e_1 & 0 & f_3 \\
0 & 0 & 0 & e_0 & -f_1 & -f_2 & -f_3 & 0 \\
0 & -f_3 & f_2 & e_1 & f_0 & 0 & 0 & 0 \\
f_3 & 0 & -f_1 & e_2 & 0 & f_0 & 0 & 0 \\
-f_2 & f_1 & 0 & e_3 & 0 & 0 & f_0 & 0 \\
-e_1 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & f_0
\end{bmatrix}
\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{bmatrix} = 0, \quad (56)$$

where

$$\begin{aligned}
e_0 &= \sum_{j=0}^3 B_{0j}(c_j + d_j), & f_0 &= c_0 - d_0, \\
e_1 &= \sum_{j=0}^3 B_{1j}(c_j + d_j), & f_1 &= c_1 - d_1, \\
e_2 &= \sum_{j=0}^3 B_{2j}(c_j + d_j), & f_2 &= c_2 - d_2, \\
e_3 &= \sum_{j=0}^3 B_{3j}(c_j + d_j), & f_3 &= c_3 - d_3.
\end{aligned} \quad (56-1)$$

$$\begin{bmatrix}
e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_4 & 0 & e_3 & -e_2 & f_1 \\
0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & -e_3 & 0 & -e_1 & -f_2 \\
0 & 0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & -e_2 & -e_1 & 0 & f_3 \\
0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & 0 & f_1 & -f_2 & -f_3 & 0 \\
0 & 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_3 & -e_2 & -e_1 & 0 & 0 & 0 & -f_4 \\
0 & 0 & 0 & 0 & 0 & e_0 & 0 & 0 & e_3 & 0 & f_1 & -f_2 & 0 & 0 & f_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e_0 & 0 & -e_2 & -f_1 & 0 & -f_3 & 0 & -f_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & -e_1 & f_2 & f_3 & 0 & f_4 & 0 & 0 & 0 \\
0 & 0 & 0 & -f_4 & 0 & -f_3 & f_2 & f_1 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -f_4 & 0 & -f_3 & 0 & e_1 & -e_2 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -f_4 & 0 & 0 & f_2 & -e_1 & 0 & -e_3 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 \\
f_4 & 0 & 0 & 0 & f_1 & e_2 & e_3 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 \\
0 & f_3 & f_2 & -e_1 & 0 & 0 & 0 & -e_4 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 \\
-f_3 & 0 & f_1 & e_2 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 \\
f_2 & f_1 & 0 & e_3 & 0 & -e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 \\
-e_1 & e_2 & -e_3 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0
\end{bmatrix}
\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{15} \\ m_{16} \end{bmatrix} = 0 \quad (57)$$

where

$$\begin{aligned}
e_0 &= \sum_{j=0}^4 B_{0j}(c_j + d_j), \quad f_0 = c_0 - d_0, \\
e_1 &= \sum_{j=0}^4 B_{1j}(c_j + d_j), \quad f_1 = c_1 - d_1, \\
e_2 &= \sum_{j=0}^4 B_{2j}(c_j + d_j), \quad f_2 = c_2 - d_2, \\
e_3 &= \sum_{j=0}^4 B_{3j}(c_j + d_j), \quad f_3 = c_3 - d_3, \\
e_4 &= \sum_{j=0}^4 B_{4j}(c_j + d_j), \quad f_4 = c_4 - d_4.
\end{aligned} \quad (57-1)$$

We must note that there are not the same linear transformations such as (51-1) - (51-2) (as exceptionally, they exist for (51)), for the third and the higher order equations of the form

$$\sum_{i,j,k=0}^n B_{ijk}(c_i c_j c_k - d_i d_j d_k) = 0, \quad \sum_{i,j,k,l=0}^n B_{ijkl}(c_i c_j c_k c_l - d_i d_j d_k d_l) = 0, \dots \quad (58)$$

Now, by the following choices,

$$\begin{aligned}
B &= \begin{bmatrix} B_{00} & B_{01} & B_{02} & \cdot & \cdot & \cdot & B_{0n} \\ B_{10} & B_{11} & B_{12} & \cdot & \cdot & \cdot & B_{1n} \\ B_{20} & B_{21} & B_{22} & \cdot & \cdot & \cdot & B_{2n} \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ B_{n0} & B_{n1} & B_{n2} & \cdot & \cdot & \cdot & B_{nn} \end{bmatrix}, \\
C &= \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}, \quad D = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix}, \quad E = \begin{bmatrix} e_0 \\ e_1 \\ e_3 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{bmatrix}, \quad F = \begin{bmatrix} f_0 \\ f_1 \\ f_3 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}; \tag{59}
\end{aligned}$$

we can rewrite the transformations (52-1) and (52-2) as follows

$$E = B(C + D), \quad F = C - D, \tag{60}$$

from (60) we also get

$$C = \frac{1}{2}(B^{-1}E + F), \quad D = \frac{1}{2}(B^{-1}E - F). \tag{61}$$

Where B^{-1} is the inverse of B ; according to equation (51), B is a symmetric matrix; also we assume that $\det B \neq 0$. The relations (60) and (61) indicate that there is an one to one correspondence between the components of C, D and the components of E, F ; using this property, as well as the relations (60) and (61), we will determine the solutions of the system of linear equations (54) – (57), on the basis of the solutions of equations (34), (36), (37) and (38).

Utilizing the standard and specific methods of solving the systems of linear equations in the set of integers [7], respectively, we obtain the following sets of the solutions for equations (34), (36), (37) and (38), (for unknowns e_i and f_i).

Firstly, we must emphasize that the general and standard solution of the equation of the type

$$\sum_{i=1}^n a_i x_i = 0 \quad (62)$$

in the set of integers, is as follows [7, 8]:

$$x_i = a_n k_i, \quad (i=1,2,3,\dots,n-1), \quad x_n = \sum_{i=1}^{n-1} a_i k_i \quad (63)$$

where the parameters k_i are arbitrary integers, and where we supposed “ $a_n \neq 0$ ”.

Now, for the equations (34) we get the following solutions (where we supposed “ $m_2 \neq 0$ ”):

$$e_0 = k_2 m_2, \quad f_0 = k_1 m_1, \quad e_1 = k_1 m_2, \quad f_1 = -k_2 m_1 \quad (64)$$

where the parameters $k_1, k_2; m_1, m_2$ are arbitrary integers.

For the equations (36) we have (where we supposed “ $m_4 \neq 0$ ”):

$$\begin{aligned} e_0 &= k_3 m_4, \quad f_0 = k_2 m_1 - k_1 m_2, \quad e_1 = k_2 m_4, \\ f_1 &= k_1 m_3 - k_3 m_1, \quad e_2 = k_1 m_4, \quad f_2 = k_3 m_2 - k_2 m_3. \end{aligned} \quad (65)$$

where the parameters $k_1, k_2, k_3; m_1, m_2, m_3, m_4$ are arbitrary integers.

Using Remark 2., we can also get the following general solutions for the system of linear equations (36) (where we supposed “ $m_4 \neq 0, k_4 \neq 0$ ”):

$$\begin{aligned} e_0 &= k_3 m_4 - k_4 m_3, \quad f_0 = k_2 m_1 - k_1 m_2, \quad e_1 = k_2 m_4 - k_4 m_2, \\ f_1 &= k_1 m_3 - k_3 m_1, \quad e_2 = k_1 m_4 - k_4 m_1, \quad f_2 = k_3 m_2 - k_2 m_3. \end{aligned} \quad (66)$$

where the parameters $k_1, k_2, k_3, k_4; m_1, m_2, m_3, m_4$ are arbitrary integers.

For the equations (37), the following solutions are obtained (where we supposed “ $m_8 \neq 0$ ”, and there is also a condition for the parameters m_i (see below)):

$$e_0 = k_4 m_8, \quad f_0 = k_3 m_1 + k_2 m_2 + k_1 m_3, \quad e_1 = k_3 m_8, \quad f_1 = -k_4 m_1 + k_1 m_6 - k_2 m_7, \quad (67)$$

$$e_2 = k_2 m_8, \quad f_2 = -k_4 m_2 - k_1 m_5 + k_3 m_7, \quad e_3 = k_1 m_8, \quad f_3 = -k_4 m_3 + k_2 m_5 - k_3 m_6.$$

where the parameters k_1, k_2, k_3, k_4 are arbitrary integers and the parameters m_i should satisfy the following equation (as a necessary condition for the parameters m_i that exist in the parametric solutions (67)):

$$m_4 m_8 = -m_1 m_5 - m_2 m_6 - m_3 m_7 \quad (68)$$

Since the parameter m_4 does not exist in the solutions (67), the condition (68), easily, could be solved by the following choices:

$$\begin{aligned} m_8 &= 1, \quad m_4 = -u_1 u_5 - u_2 u_6 - u_3 u_7, \\ m_1 &= u_1, \quad m_2 = u_2, \quad m_3 = u_3, \\ m_5 &= u_5, \quad m_6 = u_6, \quad m_7 = u_7. \end{aligned} \quad (69)$$

Now using Remark 2. and the relations (69) and (67), the solutions of (37) are determined as follows

$$\begin{aligned} e_0 &= k_4 t, \quad f_0 = k_3 u_1 + k_2 u_2 + k_1 u_3, \quad e_1 = k_3 t, \quad f_1 = -k_4 u_1 + k_1 u_6 - k_2 u_7, \\ e_2 &= k_2 t, \quad f_2 = -k_4 u_2 - k_1 u_5 + k_3 u_7, \quad e_3 = k_1 t, \quad f_3 = -k_4 u_3 + k_2 u_5 - k_3 u_6. \end{aligned} \quad (70)$$

where the parameters $t, k_1, k_2, k_3, k_4; u_1, u_2, u_3, u_5, u_6, u_7$ are arbitrary integers, and $(t \neq 0)$.

Similarly, we obtain the following solutions for the matrix equation (38) (where we supposed “ $m_{16} \neq 0$ ”, and there are also five conditions for the parameters m_i (see below)):

$$\begin{aligned} e_0 &= k_5 m_{16}, \quad f_0 = k_4 m_1 - k_3 m_2 + k_2 m_3 - k_1 m_5, \quad e_1 = k_4 m_{16}, \quad f_1 = -k_5 m_1 + k_1 m_{12} - k_2 m_{14} + k_3 m_{15}, \\ e_2 &= k_3 m_{16}, \quad f_2 = k_5 m_2 + k_1 m_{11} - k_2 m_{13} - k_4 m_{15}, \quad e_3 = k_2 m_{16}, \quad f_3 = -k_5 m_3 - k_1 m_{10} + k_3 m_{13} + k_4 m_{14}, \\ e_4 &= k_1 m_{16}, \quad f_4 = k_5 m_5 + k_2 m_{10} - k_3 m_{11} - k_4 m_{12}. \end{aligned} \quad (71)$$

where the parameters k_1, k_2, k_3, k_4, k_5 are arbitrary integers and the parameters m_i should satisfy the following equations (as the necessary conditions for the parameters m_i that exist in the parametric solutions (71)):

$$\begin{aligned}
m_4 m_{16} &= m_1 m_{13} + m_2 m_{14} - m_3 m_{15}, \\
m_6 m_{16} &= m_1 m_{11} + m_2 m_{12} - m_5 m_{15}, \\
m_7 m_{16} &= -m_1 m_{10} - m_3 m_{12} + m_5 m_{14}, \\
m_8 m_{16} &= -m_2 m_{10} + m_3 m_{11} - m_5 m_{13}, \\
m_9 m_{16} &= -m_{10} m_{15} + m_{11} m_{14} - m_{12} m_{13}.
\end{aligned} \tag{72}$$

In like manner, since the parameters m_4, m_6, m_7, m_8, m_9 do not exist in the solutions (71), the conditions (72), will be solved by the following choices:

$$\begin{aligned}
m_{16} &= 1, \\
m_4 &= u_1 u_{13} + u_2 u_{14} - u_3 u_{15}, \\
m_6 &= u_1 u_{11} + u_2 u_{12} - u_5 u_{15}, \\
m_7 &= -u_1 u_{10} - u_3 u_{12} + u_5 u_{14}, \\
m_8 &= -u_2 u_{10} + u_3 u_{11} - u_5 u_{13}, \\
m_9 &= -u_{10} u_{15} + u_{11} u_{14} - u_{12} u_{13}, \\
m_1 &= u_1, \quad m_2 = u_2, \quad m_3 = u_3, \\
m_5 &= u_5, \quad m_{10} = u_{10}, \quad m_{11} = u_{11}, \\
m_{12} &= u_{12}, \quad m_{13} = u_{13}, \quad m_{14} = u_{14}, \\
m_{15} &= u_{15}.
\end{aligned} \tag{73}$$

Using the relations (71) and (73), as well as Remark 2., the solutions of (38) become as follows

$$\begin{aligned}
e_0 &= k_5 t, \quad f_0 = k u_1 - k_3 u_2 + k_2 u_3 - k_1 u_5, \quad e_1 = k_4 t, \quad f_1 = -k_5 u_1 + k_1 u_{12} - k_2 u_{14} + k_3 u_{15}, \\
e_2 &= k_3 t, \quad f_2 = k_5 u_2 + k_1 u_{11} - k_2 u_{13} - k_4 u_{15}, \quad e_3 = k_2 t, \quad f_3 = -k_5 u_3 - k_1 u_{10} + k_3 u_{13} + k_4 u_{14}, \\
e_4 &= k_1 t, \quad f_4 = k_5 u_5 + k_2 u_{10} - k_3 u_{11} - k_4 u_{12}.
\end{aligned} \tag{71-1}$$

where the parameters $t, k_1, k_2, k_3, k_4, k_5; u_1, u_2, u_3, u_5, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ are arbitrary integers, and $(t \neq 0)$.

Similarly, the parametric solution of the matrix equation (with the size 32×32) corresponding to equation,

$$\sum_{i=0}^5 e_i f_i = 0, \quad (74)$$

similar to (64), (65), (70), (71-1), will be gotten, but with sixteen additional conditions for parameters m_i , (these conditions include sixteen homogenous second order equations, that each equation contains only four terms, similar to (68) and (72)). These conditions could be solved, easily, with some specific choices for parameters m_i , similar to (69) and (73). In general, the parametric solution of the matrix equation (i.e. the system of linear equations) corresponding to the second order equation of the form

$$\sum_{i=0}^n e_i f_i = 0 \quad (75)$$

will lead to $(2^n - \frac{n(n+1)}{2} - 1)$ number of conditions for parameters m_i (including the four terms homogenous second order equations), that these conditions will be solved by some specific choices for parameters m_i , similar to the above choices (the choices (69) and (73)), and ultimately, the general solution of (75) will be obtained.

Meanwhile, the solutions (64), (65), (70), (71-1) could also be represented as follows, respectively

$$\begin{aligned} \begin{bmatrix} e_0 \\ e_1 \end{bmatrix} &= \begin{bmatrix} k_2 t \\ k_1 t \end{bmatrix}, \\ \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} &= \begin{bmatrix} 0 & u_1 \\ -u_1 & 0 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \end{bmatrix}; \end{aligned} \quad (76)$$

where we just supposed $(m_2 = t)$ and $(m_1 = u_1)$.

$$\begin{aligned} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix} &= \begin{bmatrix} k_3 t \\ k_2 t \\ k_{12} t \end{bmatrix}, \\ \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} &= \begin{bmatrix} 0 & u_1 & -u_2 \\ -u_1 & 0 & u_3 \\ u_2 & -u_3 & 0 \end{bmatrix} \begin{bmatrix} k_3 \\ k_2 \\ k_1 \end{bmatrix}; \end{aligned} \quad (77)$$

where we just supposed $(m_4 = t)$ and $(m_1 = u_1, m_2 = u_2, m_3 = u_3)$.

$$\begin{aligned}
\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} &= \begin{bmatrix} k_4 t \\ k_3 t \\ k_2 t \\ k_1 t \end{bmatrix}, \\
\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} &= \begin{bmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & -u_7 & u_6 \\ -u_2 & u_7 & 0 & -u_5 \\ -u_3 & -u_6 & u_5 & 0 \end{bmatrix} \begin{bmatrix} k_4 \\ k_3 \\ k_2 \\ k_1 \end{bmatrix};
\end{aligned} \tag{78}$$

$$\begin{aligned}
\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} &= \begin{bmatrix} k_5 t \\ k_4 t \\ k_3 t \\ k_2 t \\ k_1 t \end{bmatrix}, \\
\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} &= \begin{bmatrix} 0 & u_1 & -u_2 & u_3 & -u_5 \\ -u_1 & 0 & u_{15} & -u_{14} & u_{12} \\ u_2 & -u_{15} & 0 & -u_{13} & u_{11} \\ -u_3 & u_{14} & u_{13} & 0 & -u_{10} \\ u_5 & -u_{12} & -u_{11} & u_{10} & 0 \end{bmatrix} \begin{bmatrix} k_5 \\ k_4 \\ k_3 \\ k_2 \\ k_1 \end{bmatrix}.
\end{aligned} \tag{79}$$

Corollary 1. If we define the matrix K as

$$K = \begin{bmatrix} k_n \\ \cdot \\ \cdot \\ \cdot \\ k_3 \\ k_2 \\ k_1 \end{bmatrix} \tag{80}$$

where we suppose $K \neq 0$, then using (60) and (61) we get the following sets of the relations

$$\begin{aligned}
C &= \frac{1}{2}(tB^{-1} + U)K, \\
D &= \frac{1}{2}(tB^{-1} - U)K, \\
E &= tK, \quad F = UK, \\
K &\neq 0, \quad \det B \neq 0;
\end{aligned} \tag{81}$$

$$\begin{aligned}
D &= (tB^{-1} - U)(tB^{-1} + U)^{-1}C, \\
E &= 2t(tB^{-1} + U)^{-1}C, \\
F &= 2M(tB^{-1} + U)^{-1}C, \quad , \\
K &= 2(tB^{-1} + U)^{-1}C, \\
C &\neq 0, \quad \det B \neq 0;
\end{aligned} \tag{82}$$

$$\begin{aligned}
C &= (tB^{-1} + U)(tB^{-1} - U)^{-1}D, \\
E &= 2t(tB^{-1} - U)^{-1}D, \\
F &= 2M(tB^{-1} - U)^{-1}D, \\
K &= 2(tB^{-1} - U)^{-1}D, \\
D &\neq 0, \quad \det B \neq 0.
\end{aligned} \tag{83}$$

In point of fact, the formulas (81) are the (integer) parametric solution of equation (51), where the matrices B, C, D, K, U have been defined by the relations (59) and (76) – (80), and the parameter t is an arbitrary integer ($t \neq 0$). On the other hand, the formulas (82) and (83) show that, if we suppose that the

values c_i (or d_i), (where $i=1,2,3,\dots,n$) are given values, then we can calculate the values d_i (or c_i) in terms of them and the matrices B, U .

Remark 3. The condition (68) (of the parameters m_i) concerning the parametric solutions (67), and the conditions (72) (of the parameters m_i) concerning the parametric solutions (71), could be solved by other methods as well.

Since the parameter m_4 does not exist in the solutions (67), the condition (68), easily will be solved by the following choices (for the parameters m_i):

$$m_4 = 0, \quad (84-1)$$

$$m_8 : \text{ a free Integer parameter } (m_8 \neq 0), \quad (84-2)$$

$$m_1 m_5 + m_2 m_6 + m_3 m_7 = 0. \quad (84-3)$$

where equation (84-3), according to the solutions (65) and (66), also could be solved as follows (including two sorts of the solutions):

$$\begin{aligned} m_1 &= u_3 v_4, \quad m_2 = u_2 v_4, \quad m_3 = u_1 v_4, \\ m_5 &= u_2 v_1 - u_1 v_2, \quad m_6 = u_1 v_3 - u_3 v_1, \quad m_7 = u_3 v_2 - u_2 v_3; \\ m_4 &= 0, \quad m_8 : \text{ a free Integer parameter } (m_8 \neq 0); \end{aligned} \quad (85)$$

and

$$\begin{aligned} m_1 &= u_3 v_4 - u_4 v_3, \quad m_2 = u_2 v_4 - u_4 v_2, \quad m_3 = u_1 v_4 - u_4 v_1, \\ m_5 &= u_2 v_1 - u_1 v_2, \quad m_6 = u_1 v_3 - u_3 v_1, \quad m_7 = u_3 v_2 - u_2 v_3, \quad m_4 = 0, \\ m_8 &: \text{ a free Integer parameter } (m_8 \neq 0). \end{aligned} \quad (86)$$

where the parameters $u_1, u_2, u_3, u_4; v_1, v_2, v_3, v_4$ are arbitrary integers. By replacing the values of m_i , (from the relations (85) or (86)) in formulas (67), as well as taking into account formulas (84-1) and (84-2), we get a new general parametric solution for the matrix equation (37).

As another similar case, since the parameters m_4, m_6, m_7, m_8, m_9 do not exist in the parametric solutions (71), the conditions (72) (for the parameters m_i , existed in the solutions (71) of the matrix equation (38)) could be solved by the following choices, as well

$$m_4 = m_6 = m_7 = m_8 = m_9 = 0, \quad (87-1)$$

$$m_{16} : \text{ a free Integer parameter } (m_{16} \neq 0), \quad (87-2)$$

$$m_1 m_{10} + m_3 m_{12} - m_5 m_{14} = 0, \quad (88-1)$$

$$m_1 m_{11} + m_2 m_{12} - m_5 m_{15} = 0, \quad (88-2)$$

$$m_1 m_{13} + m_2 m_{14} - m_3 m_{15} = 0, \quad (88-3)$$

$$m_2 m_{10} + m_5 m_{13} - m_3 m_{11} = 0, \quad (88-4)$$

$$m_{10} m_{15} + m_{12} m_{13} - m_{11} m_{14} = 0. \quad (88-5)$$

As the equation (88-5) could be derived from (88-1), (88-2), (88-3) and (88-4), we will not consider it in the next relevant calculations.

Referring to the solutions (65) and (66) (for equation (36) that corresponds to the quadratic equation (29)), respectively, the equations (88-1), (88-2), (88-3) and (88-4) are solved as follows,

first, using the solutions (65) we get:

$$m_1 = u_4 v_5, \quad m_{13} = u_3 v_2 - u_2 v_3,$$

$$(88-1) \rightarrow m_2 = u_3 v_5, \quad m_{14} = u_2 v_4 - u_4 v_2, \quad (89-1)$$

$$-m_3 = u_2 v_5, \quad m_{15} = u_4 v_3 - u_3 v_4;$$

$$\begin{aligned}
(88-2) \rightarrow \quad & m_1 = u_4 v_5, \quad m_{11} = u_3 v_1 - u_1 v_3, \\
& m_2 = u_3 v_5, \quad m_{12} = u_1 v_4 - u_4 v_1, \quad (89-2) \\
& -m_5 = u_1 v_5, \quad m_{15} = u_4 v_3 - u_3 v_4;
\end{aligned}$$

$$\begin{aligned}
(88-3) \rightarrow \quad & m_1 = u_4 v_5, \quad m_{10} = (-u_2) v_1 - u_1 v'_2, \\
& m_3 = -u_2 v_5, \quad m_{12} = u_1 v_4 - u_4 v_1, \quad (89-3) \\
& -m_5 = u_1 v_5, \quad m_{14} = u_4 v'_2 - (-u_2) v_4;
\end{aligned}$$

$$\begin{aligned}
(88-4) \rightarrow \quad & m_2 = u_3 v_5, \quad m_{10} = u_2 v'_1 - (-u_1) v_2, \\
& -m_3 = u_2 v_5, \quad m_{11} = (-u_1) v_3 - u_3 v'_1, \quad (89-4) \\
& m_5 = -u_1 v_5, \quad m_{13} = u_3 v_2 - u_2 v_3.
\end{aligned}$$

By the definitions of the type $v'_1 = -v_1$, $v'_2 = -v_2$, the solutions (89-1) – (89-4) could be simplified, for instance.

Now using the solutions (66), we get the other type (more expanded one than (89-1) – (89-4)) of the solutions for equations (88-1) – (88-4), respectively,

$$\begin{aligned}
(88-1) \rightarrow \quad & m_1 = u_4 v_5 - u_5 v_4, \quad m_{13} = u_3 v_2 - u_2 v_3, \\
& m_2 = u_3 v_5 - u_5 v_3, \quad m_{14} = u_2 v_4 - u_4 v_2, \quad (90) \\
& -m_3 = u_2 v_5 - u_5 v_2, \quad m_{15} = u_4 v_3 - u_3 v_4;
\end{aligned}$$

$$\begin{aligned}
& m_1 = u_4 v_5 - u_5 v_4, \quad m_{11} = u_3 v_1 - u_1 v_3, \\
(88-2) \rightarrow & m_2 = u_3 v_5 - u_5 v_3, \quad m_{12} = u_1 v_4 - u_4 v_1, \\
& -m_5 = u_1 v_5 - u_5 v_1, \quad m_{15} = u_4 v_3 - u_3 v_4;
\end{aligned} \tag{91}$$

$$\begin{aligned}
& m_1 = u_4 v_5 - u_5 v_4, \quad m_{10} = (-u_2) v_1 - u_1 (-v_2), \\
(88-3) \rightarrow & m_3 = -(u_2 v_5 - u_5 v_2), \quad m_{12} = u_1 v_4 - u_4 v_1, \\
& -m_5 = u_1 v_5 - u_5 v_1, \quad m_{14} = u_4 (-v_2) - (-u_2) v_4;
\end{aligned} \tag{92}$$

$$\begin{aligned}
& m_2 = u_3 v_5 - u_5 v_3, \quad m_{10} = u_2 (-v_1) - (-u_1) v_2, \\
(88-4) \rightarrow & -m_3 = u_2 v_5 - u_5 v_2, \quad m_{11} = (-u_1) v_3 - u_3 (-v_1), \\
& m_5 = -(u_1 v_5 - u_5 v_1), \quad m_{13} = u_3 v_2 - u_2 v_3.
\end{aligned} \tag{93}$$

By simplifying the relations (90) – (93), ultimately we get the following set of the general solutions for equations (88-1) – (88-4):

$$m_1 = u_4 v_5 - u_5 v_4, \quad m_2 = u_3 v_5 - u_5 v_3,$$

$$m_3 = u_5 v_2 - u_2 v_5, \quad m_4 = 0,$$

$$m_5 = u_5 v_1 - u_1 v_5, \quad m_6 = 0,$$

$$m_7 = 0, \quad m_8 = 0, \quad m_9 = 0,$$

$$m_{10} = u_1 v_2 - u_2 v_1, \quad m_{11} = u_3 v_1 - u_1 v_3,$$

$$m_{12} = u_1 v_4 - u_4 v_1, \quad m_{13} = u_3 v_2 - u_2 v_3,$$

$$m_{14} = u_2 v_4 - u_4 v_2, \quad m_{15} = u_4 v_3 - u_3 v_4,$$

$$m_{16}: \text{ a free Integer parameter } (m_{16} \neq 0). \quad (94)$$

where the parameters $u_1, u_2, u_3, u_4, u_5; v_1, v_2, v_3, v_4, v_5$ are arbitrary integers. By replacing the values of m_i (from the relations (94), in formulas (71)), as well as taking into account formulas (87-1) and (87-2), we get another form of the general parametric solution for the matrix equation (38).

Similarly, for the systems of linear equations corresponding to (75), with more variable elements (i.e. larger values of n in (75)), the similar conditions and relations to (84-1) – (84-3) and (87-1) – (87-2) and (88-1) – (88-5) and so on, could be chosen, ultimately. Then, based on them, we can determine the solution for the system of linear equations corresponding to each specific case of the quadratic equation (75).

3. Deriving the General (and Unique) Structure of the Laws Governing the Fundamental Forces of Nature.

If we assume that in the special relativity condition, the components of the n -momentum are discrete, i.e. have integer values in the invariant and the energy-momentum relations

$$g^{\mu\nu} p_\mu p_\nu = g^{\mu\nu} p'_\mu p'_\nu, \quad (95)$$

$$g^{\mu\nu} p_\mu p_\nu = (-m_0 c)^2 = g^{00} \left(\frac{-m_0 c}{\sqrt{g^{00}}} \right)^2 \quad (96)$$

where $g^{\mu\nu}$ are some constant coefficients, and p_μ, p'_μ are the components of the momentum vector in two reference frames, then the relations (95) and (96) are the special cases of the relation (51). Hence, using the matrix relations (53) – (57), we get the following systems, respectively, where they correspond to (95) and (96) (s_i are the integer parameters similar to the parameters m_i),

for (95) we have:

$$[g^{00}(p_0 + p'_0)] [s_1] = 0 \quad (97)$$

$$\begin{bmatrix} g^{0\nu}(p_\nu + p'_\nu) & p_1 - p'_1 \\ -g^{1\nu}(p_\nu + p'_\nu) & p_0 - p'_0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = 0 \quad (98)$$

where $\nu = 0, 1$;

$$\begin{bmatrix} g^{0\nu}(p_\nu + p'_\nu) & 0 & -g^{2\nu}(p_\nu + p'_\nu) & p_1 - p'_1 \\ 0 & g^{0\nu}(p_\nu + p'_\nu) & -g^{1\nu}(p_\nu + p'_\nu) & -(p_2 - p'_2) \\ p_2 - p'_2 & p_1 - p'_1 & p_0 - p'_0 & 0 \\ -g^{1\nu}(p_\nu + p'_\nu) & g^{2\nu}(p_\nu + p'_\nu) & 0 & p_0 - p'_0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 0 \quad (99)$$

where $\nu = 0, 1, 2$;

$$\begin{bmatrix} e_0 & 0 & 0 & 0 & 0 & -e_3 & e_2 & f_1 \\ 0 & e_0 & 0 & 0 & e_3 & 0 & -e_1 & f_2 \\ 0 & 0 & e_0 & 0 & -e_2 & e_1 & 0 & f_3 \\ 0 & 0 & 0 & e_0 & -f_1 & -f_2 & -f_3 & 0 \\ 0 & -f_3 & f_2 & e_1 & f_0 & 0 & 0 & 0 \\ f_3 & 0 & -f_1 & e_2 & 0 & f_0 & 0 & 0 \\ -f_2 & f_1 & 0 & e_3 & 0 & 0 & f_0 & 0 \\ -e_1 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & f_0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \end{bmatrix} = 0 \quad (100)$$

where we have

$$s_4 s_8 + s_1 s_5 + s_2 s_6 + s_3 s_7 = 0, \quad (100-1)$$

$$e_0 = g^{0\nu}(p_\nu + p'_\nu), \quad f_0 = p_0 - p'_0,$$

$$e_1 = g^{1\nu}(p_\nu + p'_\nu), \quad f_1 = p_1 - p'_1,$$

(100-2)

$$e_2 = g^{2\nu}(p_\nu + p'_\nu), \quad f_2 = p_2 - p'_2,$$

$$e_3 = g^{3\nu}(p_\nu + p'_\nu), \quad f_3 = p_3 - p'_3.$$

$$\begin{bmatrix} e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_4 & 0 & e_3 & -e_2 & f_1 \\ 0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & -e_3 & 0 & -e_1 & -f_2 \\ 0 & 0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & -e_2 & -e_1 & 0 & f_3 \\ 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & 0 & f_1 & -f_2 & -f_3 & 0 \\ 0 & 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_3 & -e_2 & -e_1 & 0 & 0 & 0 & -f_4 \\ 0 & 0 & 0 & 0 & 0 & e_0 & 0 & 0 & e_3 & 0 & f_1 & -f_2 & 0 & 0 & f_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0 & 0 & -e_2 & -f_1 & 0 & -f_3 & 0 & -f_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & -e_1 & f_2 & f_3 & 0 & f_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -f_4 & 0 & -f_3 & f_2 & f_1 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f_4 & 0 & -f_3 & 0 & e_1 & -e_2 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -f_4 & 0 & 0 & f_2 & -e_1 & 0 & -e_3 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 \\ f_4 & 0 & 0 & 0 & f_1 & e_2 & e_3 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 \\ 0 & f_3 & f_2 & -e_1 & 0 & 0 & 0 & -e_4 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 \\ -f_3 & 0 & f_1 & e_2 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 \\ f_2 & f_1 & 0 & e_3 & 0 & -e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 \\ -e_1 & e_2 & -e_3 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \\ s_{10} \\ s_{11} \\ s_{12} \\ s_{13} \\ s_{14} \\ s_{15} \\ s_{16} \end{bmatrix} = 0 \quad (101)$$

where we have

$$s_4 s_{16} = s_1 s_{13} + s_2 s_{14} - s_3 s_{15}, \quad (101-1)$$

$$s_6 s_{16} = s_1 s_{11} + s_2 s_{12} - s_5 s_{15}, \quad (101-2)$$

$$s_7 s_{16} = -s_1 s_{10} - s_3 s_{12} + s_5 s_{14}, \quad (101-3)$$

$$s_8 s_{16} = -s_2 s_{10} + s_3 s_{11} - s_5 s_{13}, \quad (101-4)$$

$$s_9 s_{16} = -s_{10} s_{15} + s_{11} s_{14} - s_{12} s_{13}; \quad (101-5)$$

$$e_0 = g^{0\nu}(p_\nu + p'_\nu), \quad f_0 = p_0 - p'_0,$$

$$e_1 = g^{1\nu}(p_\nu + p'_\nu), \quad f_1 = p_1 - p'_1,$$

$$e_2 = g^{2\nu}(p_\nu + p'_\nu), \quad f_2 = p_2 - p'_2, \quad (101-6)$$

$$e_3 = g^{3\nu}(p_\nu + p'_\nu), \quad f_3 = p_3 - p'_3,$$

$$e_4 = g^{4\nu}(p_\nu + p'_\nu), \quad f_4 = p_4 - p'_4.$$

And for (96) we get (where we suppose $\mu \neq 0$: $p'_\mu = 0$, $p'_0 = -\frac{m_0 c}{\sqrt{g^{00}}}$), respectively

$$\left[g^{00} \left(p_0 - \frac{m_0 c}{\sqrt{g^{00}}} \right) \right] [s_1] = 0 \quad (102)$$

$$\begin{bmatrix} g^{0\nu} p_\nu - g^{00} \left(\frac{m_0 c}{\sqrt{g^{00}}} \right) & p_1 \\ -g^{1\nu} p_\nu & p_0 + \left(\frac{m_0 c}{\sqrt{g^{00}}} \right) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = 0 \quad (103)$$

where $\nu = 0, 1$;

$$\begin{bmatrix} g^{0\nu} p_\nu - g^{00}(\frac{m_0 c}{\sqrt{g^{00}}}) & 0 & -g^{2\nu} p_\nu & p_1 \\ 0 & g^{0\nu} p_\nu - g^{00}(\frac{m_0 c}{\sqrt{g^{00}}}) & -g^{1\nu} p_\nu & -p_2 \\ p_2 & p_1 & p_0 + (\frac{m_0 c}{\sqrt{g^{00}}}) & 0 \\ -g^{1\nu} p_\nu & g^{2\nu} p_\nu & 0 & p_0 + (\frac{m_0 c}{\sqrt{g^{00}}}) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 0 \quad (104)$$

where $\nu = 0, 1, 2$;

$$\begin{bmatrix} e_0 & 0 & 0 & 0 & 0 & -e_3 & e_2 & f_1 \\ 0 & e_0 & 0 & 0 & e_3 & 0 & -e_1 & f_2 \\ 0 & 0 & e_0 & 0 & -e_2 & e_1 & 0 & f_3 \\ 0 & 0 & 0 & e_0 & -f_1 & -f_2 & -f_3 & 0 \\ 0 & -f_3 & f_2 & e_1 & f_0 & 0 & 0 & 0 \\ f_3 & 0 & -f_1 & e_2 & 0 & f_0 & 0 & 0 \\ -f_2 & f_1 & 0 & e_3 & 0 & 0 & f_0 & 0 \\ -e_1 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & f_0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \end{bmatrix} = 0 \quad (105)$$

where we have

$$s_4 s_8 + s_1 s_5 + s_2 s_6 + s_3 s_7 = 0 \quad (105-1)$$

$$\begin{aligned} e_0 &= g^{0\nu} p_\nu - g^{00}(\frac{m_0 c}{\sqrt{g^{00}}}), \quad f_0 = p_0 + (\frac{m_0 c}{\sqrt{g^{00}}}), \\ e_1 &= g^{1\nu} p_\nu, \quad f_1 = p_1, \\ e_2 &= g^{2\nu} p_\nu, \quad f_2 = p_2, \\ e_3 &= g^{3\nu} p_\nu, \quad f_3 = p_3. \end{aligned} \quad (105-2)$$

$$\begin{bmatrix}
e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_4 & 0 & e_3 & -e_2 & f_1 & s_1 \\
0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & -e_3 & 0 & -e_1 & -f_2 & s_2 \\
0 & 0 & e_0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & -e_2 & -e_1 & 0 & f_3 & s_3 \\
0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 & 0 & f_1 & -f_2 & -f_3 & 0 & s_4 \\
0 & 0 & 0 & 0 & e_0 & 0 & 0 & 0 & 0 & e_3 & -e_2 & -e_1 & 0 & 0 & 0 & -f_4 & s_5 \\
0 & 0 & 0 & 0 & 0 & e_0 & 0 & 0 & e_3 & 0 & f_1 & -f_2 & 0 & 0 & f_4 & 0 & s_6 \\
0 & 0 & 0 & 0 & 0 & 0 & e_0 & 0 & -e_2 & -f_1 & 0 & -f_3 & 0 & -f_4 & 0 & 0 & s_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e_0 & -e_1 & f_2 & f_3 & 0 & f_4 & 0 & 0 & 0 & s_8 \\
0 & 0 & 0 & -f_4 & 0 & -f_3 & f_2 & f_1 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_9 \\
0 & 0 & -f_4 & 0 & -f_3 & 0 & e_1 & -e_2 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & 0 & s_{10} \\
0 & -f_4 & 0 & 0 & f_2 & -e_1 & 0 & -e_3 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & 0 & s_{11} \\
f_4 & 0 & 0 & 0 & f_1 & e_2 & e_3 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & 0 & s_{12} \\
0 & f_3 & f_2 & -e_1 & 0 & 0 & 0 & -e_4 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & s_{13} \\
-f_3 & 0 & f_1 & e_2 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & s_{14} \\
f_2 & f_1 & 0 & e_3 & 0 & -e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & s_{15} \\
-e_1 & e_2 & -e_3 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_0 & s_{16}
\end{bmatrix} = 0 \quad (106)$$

where we have

$$s_4 s_{16} = s_1 s_{13} + s_2 s_{14} - s_3 s_{15}, \quad (106-1)$$

$$s_6 s_{16} = s_1 s_{11} + s_2 s_{12} - s_5 s_{15}, \quad (106-2)$$

$$s_7 s_{16} = -s_1 s_{10} - s_3 s_{12} + s_5 s_{14}, \quad (106-3)$$

$$s_8 s_{16} = -s_2 s_{10} + s_3 s_{11} - s_5 s_{13}, \quad (106-4)$$

$$s_9 s_{16} = -s_{10} s_{15} + s_{11} s_{14} - s_{12} s_{13}; \quad (106-5)$$

$$\begin{aligned}
e_0 &= g^{0\nu} p_\nu - g^{00} \left(\frac{m_0 c}{\sqrt{g^{00}}} \right), \quad f_0 = p_0 + g^{00} \left(\frac{m_0 c}{\sqrt{g^{00}}} \right), \\
e_1 &= g^{1\nu} p_\nu, \quad f_1 = p_1, \\
e_2 &= g^{2\nu} p_\nu, \quad f_2 = p_2, \\
e_3 &= g^{3\nu} p_\nu, \quad f_3 = p_3, \\
e_4 &= g^{4\nu} p_\nu, \quad f_4 = p_4.
\end{aligned} \tag{106-6}$$

Remark 4. In this section we will use the geometrized unites [9], Einstein notation, as well as the following conventions:

$$\begin{aligned}
\text{-Metric sign convention:} \quad & (+, -, -, \dots, -), \\
\text{-Riemann and Ricci tensors:} \quad & R^\rho_{\sigma\mu\nu} = \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\sigma\mu} - \partial_\mu \Gamma^\rho_{\sigma\nu} - \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\sigma\nu}, \\
& R_{\sigma\mu} = -R^\nu_{\sigma\mu\nu}, \\
\text{-Einstein tensor:} \quad & (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = -8\pi T_{\mu\nu} + \dots \quad . \tag{107}
\end{aligned}$$

Corollary 2. It is worth to note that concerning the relations (97) – (101) for the components p_μ and p'_μ , we can use the relations (81) – (83) and the solutions (76) – (79), to determine the linear transformations between two reference frames. In other word, the general forms of the linear transformations between two reference frames, directly and immediately, are determined from the relations (97) – (101) for all dimensions.

Now, we use equations (102) – (106) for deriving the equations governing the fundamental forces of physics.

First, we consider the following definitions for the canonical operators:

$$p_\mu \rightarrow i\hbar \nabla_\mu \tag{108}$$

$$g^{\mu\nu} (\text{const.}) \rightarrow g^{\mu\nu} (\text{components of the metric tensor}) \tag{109}$$

$$s_i \rightarrow \hat{s}_i = F_{\mu\nu}, \quad Z_{\mu\nu\rho}, \dots (\text{components of the field tensor}) \tag{110}$$

We do substitute the operators (108) – (109) with the corresponding quantities in equations (102) – (106). According to the structures, the compositions and the number of equations of the systems (102) – (106), we do conclude that there exist only three kinds of anti-symmetric tensors that their components could be substituted with the parameters s_i (except in the equation (102), which is a special and trivial case), and transform the systems (102) – (106) to the tensor equations. These tensors are a 2nd order tensor, a 3rd order tensor and a 4th order tensor. The 4th order tensor of these tensors just matches the Riemann tensor $R_{\mu\nu\rho\sigma}$ (which at the same time, it is necessary for calculating and specifying the components of the metric tensor $g^{\mu\nu}$), the other two tensors could be written as $Z_{\mu\nu\rho}$ and $F_{\mu\nu}$, that should be anti-symmetric with respect to the indices μ, ν such as: $Z_{\mu\nu\rho} = -Z_{\nu\mu\rho}$, $F_{\mu\nu} = -F_{\nu\mu}$.

Thus firstly, corresponding to equations (102) – (104) (that they don't contain any condition for parameters s_i), we get the following tensor equations, respectively

(the tensor equation corresponding to relation (102) is a special and trivial case, that Riemann tensor also vanished there; in any case we just write it down, assuming a tensor such as $\tilde{F}_\mu$ substituting with s_1 and where we assume $g^{00} = 1$)

$$(102) \rightarrow D_\mu^* \tilde{F}^\mu = 0 \quad (111-1)$$

where $\mu = 0$, and $g^{00} = 1$, $s_1 \rightarrow \hat{s}_1 = \tilde{F}_0$.

$$(103) \rightarrow D_{[\rho} F_{\mu\nu]} = 0, \quad (112-1)$$

$$D_\mu^* F_\nu^\mu = -J_\nu^{(E)} \quad (112-2)$$

where $\rho, \mu, \nu = 0, 1$, and $s_1 \rightarrow \hat{s}_1 = F_{10}$, $s_2 \rightarrow \hat{s}_2 = \varphi^{(E)}$, $J_\nu^{(E)} = -D_\nu \varphi^{(E)}$.

$$(103) \rightarrow D_{[\rho} Z_{\mu\nu]\sigma} = 0, \quad (112-3)$$

$$D_\mu^* Z_{\nu\rho}^\mu = -J_{\nu\rho}^{(N)} \quad (112-4)$$

where $\rho, \sigma, \mu, \nu = 0, 1$, and $s_1 \rightarrow \hat{s}_1 = Z_{10\rho}$, $s_2 \rightarrow \hat{s}_2 = \varphi_\rho^{(N)}$, $J_{\nu\rho}^{(N)} = -D_\nu \varphi_\rho^{(N)}$.

$$(103) \rightarrow D_{[\lambda} R_{\mu \nu] \rho \sigma} = 0, \quad (112-5)$$

$$D_{\mu}^* R_{\nu \rho \sigma}^{\mu} = -J_{\nu \rho \sigma}^{(G)} \quad (112-6)$$

where $\lambda, \rho, \sigma, \mu, \nu = 0, 1$, and $s_1 \rightarrow \hat{s}_1 = R_{10 \rho \sigma}$, $s_2 \rightarrow \hat{s}_2 = \varphi_{\rho \sigma}^{(G)}$, $J_{\nu \rho \sigma}^{(G)} = -D_{\nu} \varphi_{\rho \sigma}^{(G)}$.

$$(104) \rightarrow D_{[\rho} F_{\mu \nu]} = 0, \quad (113-1)$$

$$D_{\mu}^* F_{\nu}^{\mu} = -J_{\nu}^{(E)} \quad (113-2)$$

where $\rho, \mu, \nu = 0, 1, 2$, and

$s_1 \rightarrow \hat{s}_1 = F_{10}$, $s_2 \rightarrow \hat{s}_2 = F_{02}$, $s_3 \rightarrow \hat{s}_3 = F_{21}$, $s_4 \rightarrow \hat{s}_4 = \varphi^{(E)}$, $J_{\nu}^{(E)} = -D_{\nu} \varphi^{(E)}$.

$$(104) \rightarrow D_{[\rho} Z_{\mu \nu] \sigma} = 0, \quad (113-3)$$

$$D_{\mu}^* Z_{\nu \rho}^{\mu} = -J_{\nu \rho}^{(N)} \quad (113-4)$$

where $\rho, \sigma, \mu, \nu = 0, 1, 2$, and

$s_1 \rightarrow \hat{s}_1 = Z_{10 \rho}$, $s_2 \rightarrow \hat{s}_2 = Z_{02 \rho}$, $s_3 \rightarrow \hat{s}_3 = Z_{21 \rho}$, $s_4 \rightarrow \hat{s}_4 = \varphi_{\rho}^{(N)}$, $J_{\nu \rho}^{(N)} = -D_{\nu} \varphi_{\rho}^{(N)}$.

$$(104) \rightarrow D_{[\lambda} R_{\mu \nu] \rho \sigma} = 0, \quad (113-5)$$

$$D_{\mu}^* R_{\nu \rho \sigma}^{\mu} = -J_{\nu \rho \sigma}^{(G)} \quad (113-6)$$

where $\lambda, \rho, \sigma, \mu, \nu = 0, 1, 2$, and

$$s_1 \rightarrow \hat{s}_1 = R_{10\rho\sigma}, \quad s_2 \rightarrow \hat{s}_2 = R_{02\rho\sigma}, \quad s_3 \rightarrow \hat{s}_3 = R_{21\rho\sigma}, \quad s_4 \rightarrow \hat{s}_4 = \varphi_{\rho\sigma}^{(G)}, \quad J_{\nu\rho\sigma}^{(G)} = -D_\nu \varphi_{\rho\sigma}^{(G)}.$$

where in equations (111-1) – (113-6) we have

$$D_\mu = \nabla_\mu + \frac{im_0}{\hbar} k_\mu, \quad (114-1)$$

$$D_\mu^* = \nabla_\mu - \frac{im_0}{\hbar} k_\mu. \quad (114-2)$$

$$\mu = 0: \quad k_\mu = \frac{1}{\sqrt{g^{00}}}, \quad (115-1)$$

$$\mu \neq 0: \quad k_\mu = 0,$$

$$\nabla_\nu I^{(E)\nu} = 0, \quad I_\nu^{(E)} = J_\nu^{(E)} - \frac{im_0}{\hbar} k_\mu F_\nu^\mu, \quad (116-1)$$

$$\nabla_\nu I_\rho^{(N)\nu} = 0, \quad I_{\nu\rho}^{(N)} = J_{\nu\rho}^{(N)} - \frac{im_0}{\hbar} k_\mu Z_{\nu\rho}^\mu, \quad (116-2)$$

$$\nabla_\nu I_{\rho\sigma}^{(G)\nu} = 0, \quad I_{\nu\rho\sigma}^{(G)} = J_{\nu\rho\sigma}^{(G)} - \frac{im_0}{\hbar} k_\mu R_{\nu\rho\sigma}^\mu, \quad D_\nu^* J_{\rho\sigma}^{(G)\nu} = 0. \quad (116-3)$$

In principle, we suppose and conclude that the unique tensor equations (112-1) – (113-6) (including the definitions and relations (114-1) – (116-3)) are the laws governing the fundamental forces of nature, respectively in the dimensions $D = 2, 3$.

In the system of linear equations (102) – (104), the parameters s_i were just arbitrary integer parameters, that we substituted the components of tensors $F_{\mu\nu}, Z_{\mu\nu\rho}, R_{\mu\nu\rho\sigma}$ with them (based on the principal operator definitions (108) – (110)).

Before writing the tensor equations corresponding to the relations (105) and (106), which are similar to the tensor equations (112-1) – (113-6) (as we will show), let at first mention the following remark and corollary.

Remark 5. For the next systems of linear equations, i.e. (105) and (106) (and so on), the situation for the parameters s_i is a bit different. There are some conditions for the parameters s_i (including the quadratic

equations (105-1) and (106-1) – (106-5)), that should be considered and solved. In the previous section, we dealt with this situation in two different ways. We had two types of the general solutions for these conditions; one includes the solutions of the form (69) and (73) (that simply, they respectively become the solutions of the conditions (105-1) and (106-1) – (106-5), by $m_i \rightarrow s_i$). The other one includes the solutions of the forms (85) and (86) (that simply, they become the solutions of the condition (105-1), by $m_i \rightarrow s_i$), and the solutions of the forms (89-1) – (89-4) and (94) (that simply, they become the solutions of the conditions (106-1) – (106-5), by $m_i \rightarrow s_i$). In all these solutions for the conditions (105-1) and (106-1) – (106-5), the parameters s_i , necessarily, are written in terms of the new parameters u_i and v_i , which we may represent them as the general form

$$s_i = H_i(u_1, u_2, u_3, \dots, u_m; v_1, v_2, v_3, \dots, v_n) \quad (117)$$

where the parameters u_i and v_i are arbitrary integers. Now, for dealing with this situation, in principle, we accept that:

“ Due to the tensor components (corresponding to the operators $\hat{s}_i$, according to the definitions (108) – (110)) substitute with the parameters s_i , and at the same time the parameters s_i are not arbitrary parameters and they defined by (117) (where u_i and v_i are arbitrary parameters), then we conclude that the operators $\hat{s}_i$ should have the following form, as well

$$\hat{s}_i = H_i(\hat{u}_1, \hat{u}_2, \hat{u}_3, \dots, \hat{u}_m; \hat{v}_1, \hat{v}_2, \hat{v}_3, \dots, \hat{v}_n) \quad (118)$$

where $\hat{u}_i$ and $\hat{v}_i$ are some operators that substitute with the parameters u_i and v_i , and they should be specified for the tensor components corresponding to the operators $\hat{s}_i$. So, from the above conditions and arguments, in principle, we accept and conclude that

$$[s_i = H_i(u_1, u_2, u_3, \dots, u_m; v_1, v_2, v_3, \dots, v_n)] \Leftrightarrow [\hat{s}_i = H_i(\hat{u}_1, \hat{u}_2, \hat{u}_3, \dots, \hat{u}_m; \hat{v}_1, \hat{v}_2, \hat{v}_3, \dots, \hat{v}_n)].” \quad (119)$$

Meanwhile, according to the parametric solutions (69), (73), (85), (86), (89-1) – (89-4) and (94), the form H_i is not a unique form. In Corollary 3. by using (119), the form H_i , as well as $\hat{u}_i$ and $\hat{v}_i$ could be specified (in fact, beforehand, we will clarify that which form(s) of H_i are acceptable), such that they

will be consistent with the operators $\hat{S}_i$ (that correspond to the components of the tensors $F_{\mu\nu}, Z_{\mu\nu\rho}, R_{\mu\nu\rho\sigma}$).

Corollary 3. Concerning Remark 5. and the relations (118) and (119), for specifying the form H_i , and the operators $\hat{u}_i$ and $\hat{v}_i$ in the structures of the tensors $F_{\mu\nu}, Z_{\mu\nu\rho}, R_{\mu\nu\rho\sigma}$, basically, we will use the Riemann tensor as the base, as on one hand it is necessary for specifying the components of the metric tensor $g^{\mu\nu}$, and on other hand it is a basal tensor with a “specific structure”.

Thus, on this basis, concerning the form H_i in (117), (118) and (119), only the parametric solutions of the type (86) (with respect to $m_i \rightarrow s_i$):

$$\begin{aligned}
s_1 &= u_3 v_4 - u_4 v_3, & s_2 &= u_2 v_4 - u_4 v_2, \\
s_3 &= u_1 v_4 - u_4 v_1, & s_5 &= u_2 v_1 - u_1 v_2, \\
s_6 &= u_1 v_3 - u_3 v_1, & s_7 &= u_3 v_2 - u_2 v_3, & s_4 &= 0, \\
s_8 &: \text{ a free Integer parameter } (s_8 \neq 0).
\end{aligned} \tag{117-1}$$

for equation (105-1), as well as only the parametric solutions of the type (94) (with respect to $m_i \rightarrow s_i$):

$$\begin{aligned}
s_1 &= u_4 v_5 - u_5 v_4, & s_2 &= u_3 v_5 - u_5 v_3, \\
s_3 &= u_5 v_2 - u_2 v_5, & s_4 &= 0, \\
s_5 &= u_5 v_1 - u_1 v_5, & s_6 &= 0, \\
s_7 &= 0, & s_8 &= 0, & s_9 &= 0, \\
s_{10} &= u_1 v_2 - u_2 v_1, & s_{11} &= u_3 v_1 - u_1 v_3, \\
s_{12} &= u_1 v_4 - u_4 v_1, & s_{13} &= u_3 v_2 - u_2 v_3, \\
s_{14} &= u_2 v_4 - u_4 v_2, & s_{15} &= u_4 v_3 - u_3 v_4, \\
s_{16} &: \text{ a free Integer parameter } (s_{16} \neq 0).
\end{aligned} \tag{117-2}$$

for equation (106-1) – (106-5), are acceptable; furthermore, by starting from the Riemann tensor and its specific structure in (107) and using the relation $\Gamma_{\sigma\mu}^{\lambda} = g^{\beta\lambda}\Gamma_{\beta\sigma\mu}$, we get

$$R_{\rho\sigma\mu\nu} = (\partial_{\nu}\Gamma_{\rho\sigma\mu} - \Gamma_{\rho\nu}^{\lambda}\Gamma_{\lambda\sigma\mu}) - (\partial_{\mu}\Gamma_{\rho\sigma\nu} - \Gamma_{\rho\mu}^{\lambda}\Gamma_{\lambda\sigma\nu}) \quad (120)$$

If we define the operator C_{μ} , operating on a second order tensor $Y_{\nu\rho}$ as follows

$$C_{\mu}Y_{\nu\rho} = a(\partial_{\nu}Y_{\rho\mu} + \partial_{\rho}Y_{\nu\mu} - \partial_{\mu}Y_{\nu\rho}) \quad (121)$$

where a is a non-zero constant, then we may rewrite (120) such as

$$R_{\rho\sigma\mu\nu} = \frac{1}{a}[(\partial_{\nu}C_{\rho} - \Gamma_{\rho\nu}^{\lambda}C_{\lambda})g_{\sigma\mu} - (\partial_{\mu}C_{\rho} - \Gamma_{\rho\mu}^{\lambda}C_{\lambda})g_{\sigma\nu}] \quad (122)$$

Now, we define the operator $D_{\rho\mu}$ as well

$$D_{\rho\mu} = (\partial_{\mu}C_{\rho} - \Gamma_{\rho\mu}^{\lambda}C_{\lambda}) \quad (123)$$

Then the relation (122) for the Riemann tensor could be written as follows

$$R_{\rho\sigma\mu\nu} = \frac{1}{a}(D_{\rho\nu}g_{\sigma\mu} - D_{\rho\mu}g_{\sigma\nu}) \quad (124)$$

Meanwhile, the natural generalization of the operator C_{ρ} , operating on an arbitrary n^{th} order tensor $A_{\beta_1\beta_2\beta_3...\beta_n}$, is

$$C_{\rho}A_{\beta_1\beta_2\beta_3...\beta_n} = a(\partial_{[\beta_1}A_{\beta_2\beta_3\beta_4...\beta_n]\rho} - \partial_{\rho}A_{\beta_1\beta_2\beta_3...\beta_n}) \quad (125)$$

According to (122) and (124), the only and the most general form representing the structure of the Riemann tensor, is

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = \frac{1}{a}(\hat{A}_{\rho\nu}\hat{B}_{\sigma\mu} - \hat{A}_{\rho\mu}\hat{B}_{\sigma\nu}) \quad (126)$$

where

$$\hat{A}_{\rho\mu} = D_{\rho\mu}, \quad \hat{B}_{\sigma\mu} = g_{\sigma\mu}. \quad (127)$$

Now, as stated in Corollary 3., and Remark 5., we use and apply the relation (126) (as a basal and principal criterion), as well as the relations (117) – (119), for determining the general formulation of the operators $\hat{s}_i$ (that substitute with the parameters s_i in the systems (103) – (106)), which correspond to the components of the Riemann tensor. That is, respectively

$$(103): \{(a\hat{s}_1 = aR_{10\rho\sigma} = \hat{A}_{\rho 0}\hat{B}_{\sigma 1} - \hat{A}_{\rho 1}\hat{B}_{\sigma 0}, \quad \hat{s}_2 = \varphi_{\rho\sigma}^{(G)}),$$

$$(as_1 = A_0B_1 - A_1B_0, \quad (128)$$

$$s_2 : a \text{ free Integer parameter } (s_2 \neq 0));$$

$$(104): \{(a\hat{s}_1 = aR_{10\rho\sigma} = \hat{A}_{\rho 0}\hat{B}_{\sigma 1} - \hat{A}_{\rho 1}\hat{B}_{\sigma 0}, \quad a\hat{s}_2 = aR_{02\rho\sigma} = \hat{A}_{\rho 2}\hat{B}_{\sigma 0} - \hat{A}_{\rho 0}\hat{B}_{\sigma 2},$$

$$a\hat{s}_3 = aR_{21\rho\sigma} = \hat{A}_{\rho 1}\hat{B}_{\sigma 2} - \hat{A}_{\rho 2}\hat{B}_{\sigma 1}, \quad \hat{s}_4 = \varphi_{\rho\sigma}^{(G)}), \quad (as_1 = A_0B_1 - A_1B_0, \quad (129)$$

$$as_2 = A_2B_0 - A_0B_2, \quad as_3 = A_1B_2 - A_2B_1,$$

$$s_4 : a \text{ free Integer parameter } (s_4 \neq 0));$$

$$(105): \{(a\hat{s}_1 = aR_{10\rho\sigma} = \hat{A}_{\rho 0}\hat{B}_{\sigma 1} - \hat{A}_{\rho 1}\hat{B}_{\sigma 0}, \quad a\hat{s}_2 = aR_{20\rho\sigma} = \hat{A}_{\rho 0}\hat{B}_{\sigma 2} - \hat{A}_{\rho 2}\hat{B}_{\sigma 0},$$

$$a\hat{s}_3 = aR_{30\rho\sigma} = \hat{A}_{\rho 0}\hat{B}_{\sigma 3} - \hat{A}_{\rho 3}\hat{B}_{\sigma 0}, \quad \hat{s}_4 = 0, \quad a\hat{s}_5 = aR_{23\rho\sigma} = \hat{A}_{\rho 3}\hat{B}_{\sigma 2} - \hat{A}_{\rho 2}\hat{B}_{\sigma 3},$$

$$a\hat{s}_6 = aR_{31\rho\sigma} = \hat{A}_{\rho 1}\hat{B}_{\sigma 3} - \hat{A}_{\rho 3}\hat{B}_{\sigma 1}, \quad a\hat{s}_7 = aR_{12\rho\sigma} = \hat{A}_{\rho 2}\hat{B}_{\sigma 1} - \hat{A}_{\rho 1}\hat{B}_{\sigma 2}, \quad \hat{s}_8 = \varphi_{\rho\sigma}^{(G)}), \quad (130)$$

$$(as_1 = A_0B_1 - A_1B_0, \quad as_2 = A_0B_2 - A_2B_0, \quad as_3 = A_0B_3 - A_3B_0, \quad s_4 = 0,$$

$$as_5 = A_3B_2 - A_2B_3, \quad as_6 = A_1B_3 - A_3B_1, \quad as_7 = A_2B_1 - A_1B_2,$$

$$s_8 : a \text{ free Integer parameter } (s_8 \neq 0));$$

$$\begin{aligned}
(106): \{ & (a\hat{s}_1 = aR_{10\rho\sigma} = \hat{A}_{\rho 0}\hat{B}_{\sigma 1} - \hat{A}_{\rho 1}\hat{B}_{\sigma 0}, \quad a\hat{s}_2 = aR_{02\rho\sigma} = \hat{A}_{\rho 2}\hat{B}_{\sigma 0} - \hat{A}_{\rho 0}\hat{B}_{\sigma 2}, \\
& a\hat{s}_3 = aR_{30\rho\sigma} = \hat{A}_{\rho 0}\hat{B}_{\sigma 3} - \hat{A}_{\rho 3}\hat{B}_{\sigma 0}, \quad \hat{s}_4 = 0, \quad a\hat{s}_5 = aR_{04\rho\sigma} = \hat{A}_{\rho 4}\hat{B}_{\sigma 0} - \hat{A}_{\rho 0}\hat{B}_{\sigma 4}, \\
& \hat{s}_6 = 0, \quad \hat{s}_7 = 0, \quad \hat{s}_8 = 0 \quad \hat{s}_9 = 0, \quad a\hat{s}_{10} = aR_{34\rho\sigma} = \hat{A}_{\rho 4}\hat{B}_{\sigma 3} - \hat{A}_{\rho 3}\hat{B}_{\sigma 4}, \\
& a\hat{s}_{11} = aR_{42\rho\sigma} = \hat{A}_{\rho 2}\hat{B}_{\sigma 4} - \hat{A}_{\rho 4}\hat{B}_{\sigma 2}, \quad a\hat{s}_{12} = aR_{41\rho\sigma} = \hat{A}_{\rho 1}\hat{B}_{\sigma 4} - \hat{A}_{\rho 4}\hat{B}_{\sigma 1}, \\
& a\hat{s}_{13} = aR_{23\rho\sigma} = \hat{A}_{\rho 3}\hat{B}_{\sigma 2} - \hat{A}_{\rho 2}\hat{B}_{\sigma 3}, \quad a\hat{s}_{14} = aR_{13\rho\sigma} = \hat{A}_{\rho 3}\hat{B}_{\sigma 1} - \hat{A}_{\rho 1}\hat{B}_{\sigma 3}, \\
& a\hat{s}_{15} = aR_{21\rho\sigma} = \hat{A}_{\rho 1}\hat{B}_{\sigma 2} - \hat{A}_{\rho 2}\hat{B}_{\sigma 1}, \quad \hat{s}_{16} = \varphi_{\rho\sigma}^{(G)}), \\
& (as_1 = A_0B_1 - A_1B_0, \quad as_2 = A_2B_0 - A_0B_2, \quad as_3 = A_0B_3 - A_3B_0, \\
& s_4 = 0, \quad as_5 = A_4B_0 - A_0B_4, \quad s_6 = 0, \quad s_7 = 0, \quad s_8 = 0, \quad s_9 = 0, \\
& as_{10} = A_4B_3 - A_3B_4, \quad as_{11} = A_2B_4 - A_4B_2, \quad as_{12} = A_1B_4 - A_4B_1, \\
& as_{13} = A_3B_2 - A_2B_3, \quad as_{14} = A_3B_1 - A_1B_3, \quad as_{15} = A_1B_2 - A_2B_1, \\
& s_{16} : a \text{ free Integer parameter } (s_{16} \neq 0))\}.
\end{aligned} \tag{131}$$

where the parameters A_i and B_i are arbitrary. In the above formulas for s_i , the element a is just an arbitrary parameter ($a \neq 0$). But in the above formulas representing $\hat{s}_i$, the element a will be specified later (that is $a = i\hbar$).

As we stated in **Corollary 3.**, as well as according to formulas (128) – (131), it is clear that for the form H_i in (117), (118) and (119), only the relations (117-1) and (117-2) are acceptable, which are consistent with the formulas (126) and (128) – (131). Firstly, it is easy to show that the algebraic formulas representing s_i in (128) and (129), are also consistent with previous algebraic conditions of the parameters s_i in the systems (103) and (104) (where they just were arbitrary integers). For example, regarding the algebraic formulas representing s_i in (129), by the following choices

$$A_0 = -(k_1 s'_2 + k_2 s'_1), \quad B_0 = -(k'_1 s'_2 + k'_2 s'_1), \quad a = (k_1 k'_2 - k_2 k'_1) s'_3, \quad (132)$$

$$A_1 = k_1 s'_3, \quad B_1 = k'_1 s'_3, \quad A_2 = k_2 s'_3, \quad B_2 = k'_2 s'_3.$$

where the parameters k_i, k'_i, s'_i are arbitrary integers, directly we can show that the parameters s_i are also arbitrary and each could take any integer value.

Secondly, concerning the formulas (130) and (131), directly we could show that by the following choices, respectively,

$$\begin{aligned} A_0 &= av_4, \quad B_0 = u_4, \quad A_1 = av_3, \quad B_1 = u_3, \\ A_2 &= av_2, \quad B_2 = u_2, \quad A_3 = av_1, \quad B_3 = u_1, \end{aligned} \quad (133)$$

$$s_4 = 0, \quad s_8 : a \text{ free Integer parameter } (s_8 \neq 0);$$

$$\begin{aligned} A_0 &= av_5, \quad B_0 = u_5, \quad A_1 = av_4, \quad B_1 = u_4, \quad A_2 = -av_3, \\ B_2 &= -u_3, \quad A_3 = -av_2, \quad B_3 = -u_2, \quad A_4 = av_1, \quad B_4 = u_1, \end{aligned} \quad (134)$$

$$s_4 = 0, \quad s_6 = 0, \quad s_7 = 0, \quad s_8 = 0, \quad s_9 = 0,$$

$$s_{16} : a \text{ free Integer parameter } (s_{16} \neq 0)$$

the formulas (130) and (131) completely accord with formulas (117-1) and (117-2) (and only with them, as it should be).

Now, according to the above suppositions and approaches, such as Remark 5. and Corollary 3. and the relations (130) and (133), we derive the following unique field equations as the laws governing the fundamental forces of nature in the fourth dimension, respectively

$$(105) \rightarrow D_{[\rho} F_{\mu \nu]} = 0, \quad (135-1)$$

$$D_\mu^* F_\nu^\mu = -J_\nu^{(E)} \quad (135-2)$$

where $\rho, \mu, \nu = 0, 1, 2, 3$ and

$$s_1 \rightarrow \hat{s}_1 = F_{10}, \quad s_2 \rightarrow \hat{s}_2 = F_{20}, \quad s_3 \rightarrow \hat{s}_3 = F_{30},$$

$$s_4 \rightarrow \hat{s}_4 = 0, \quad s_5 \rightarrow \hat{s}_5 = F_{23}, \quad s_6 \rightarrow \hat{s}_6 = F_{31},$$

$$s_7 \rightarrow \hat{s}_7 = F_{12}, \quad s_8 \rightarrow \hat{s}_8 = \varphi^{(E)}, \quad J_\nu^{(E)} = -D_\nu \varphi^{(E)}.$$

$$(105) \rightarrow D_{[\rho} Z_{\mu \nu] \sigma} = 0, \quad (135-3)$$

$$D_\mu^* Z_{\nu \rho}^\mu = -J_{\nu \rho}^{(N)} \quad (135-4)$$

where $\rho, \sigma, \mu, \nu = 0, 1, 2, 3$ and

$$s_1 \rightarrow \hat{s}_1 = Z_{10\rho}, \quad s_2 \rightarrow \hat{s}_2 = Z_{20\rho}, \quad s_3 \rightarrow \hat{s}_3 = Z_{30\rho},$$

$$s_4 \rightarrow \hat{s}_4 = 0, \quad s_5 \rightarrow \hat{s}_5 = Z_{23\rho}, \quad s_6 \rightarrow \hat{s}_6 = Z_{31\rho},$$

$$s_7 \rightarrow \hat{s}_1 = Z_{12\rho}, \quad s_8 \rightarrow \hat{s}_8 = \varphi_\rho^{(N)}, \quad J_{\nu \rho}^{(N)} = -D_\nu \varphi_\rho^{(N)}.$$

$$(105) \rightarrow D_{[\lambda} R_{\mu \nu] \rho \sigma} = 0, \quad (135-5)$$

$$D_\mu^* R_{\nu \rho \sigma}^\mu = -J_{\nu \rho \sigma}^{(G)} \quad (135-6)$$

Where $\lambda, \rho, \sigma, \mu, \nu = 0, 1, 2, 3$ and

$$s_1 \rightarrow \hat{s}_1 = R_{10\rho\sigma}, \quad s_2 \rightarrow \hat{s}_2 = R_{20\rho\sigma}, \quad s_3 \rightarrow \hat{s}_3 = R_{30\rho\sigma},$$

$$s_4 \rightarrow \hat{s}_4 = 0, \quad s_5 \rightarrow \hat{s}_5 = R_{23\rho\sigma}, \quad s_6 \rightarrow \hat{s}_6 = R_{31\rho\sigma},$$

$$s_7 \rightarrow \hat{s}_7 = R_{12\rho\sigma}, \quad s_8 \rightarrow \hat{s}_8 = \varphi_{\rho\sigma}^{(G)}, \quad J_{\nu\rho\sigma}^{(G)} = -D_\nu \varphi_{\rho\sigma}^{(G)}.$$

Where the definitions and relations (114-1) – (116-3) are applied to the equations (135-1) – (135-6) as well¹.

Corollary 4. Now, as stated in Corollary 3. and Remark 5., according to the relations (117) – (119) and (123), (125), (127) and (128) – (131), as well as (126) (as a basal and principal criterions for the specifying the structures of the quantities $\hat{s}_i$, that correspond to the components of the field tensors), the following relations regarding the structures of the field tensors $F_{\mu\nu}$ (the representer of the electromagnetic field) and $Z_{\mu\nu\rho}$ (the representer of the strong nuclear field, as it should be) in all equations (112-1) – (113-6) and (135-1) – (135-6), are necessary, respectively

$$F_{\mu\nu} = \frac{1}{a} (D_{\mu\nu} Q - D_{\nu\mu} Q) \quad (136)$$

$$Z_{\mu\nu\rho} = S_{\rho\mu\nu} = \frac{1}{a} (D_{\rho\nu} H_\mu - D_{\rho\mu} H_\nu) \quad (137)$$

where Q and H_μ are a scalar field and a vector field, and where we have $a = i\hbar$, as well as the following gauge operators that ultimately, are obtained for the fields $F_{\mu\nu}$ and $Z_{\mu\nu\rho}$ (or $S_{\rho\mu\nu}$):

$$F_{\mu\nu}: \quad \nabla_\mu \rightarrow \nabla_\mu + \frac{ig^{(E)}}{\hbar} A_\mu, \quad (138-1)$$

$$A_\mu = C_\mu Q = -\partial_\mu Q,$$

$$F_{\mu\nu} = \nabla_\nu A_\mu - \nabla_\mu A_\nu + \frac{ig^{(E)}}{\hbar} [A_\nu, A_\mu]. \quad (138-2)$$

where $g^{(E)}$ is the coupling constant;

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1. Similarly, using the relations (131), (134) and (108) – (110), we may obtain the tensor equations corresponding to the system (106). However, there might be appeared some limitations for the higher dimensions if we suppose the discreteness of the other quantities such as space-time coordinates and so on.

and

$$Z_{\mu\nu\rho} : \quad \nabla_\mu \rightarrow \nabla_\mu + \frac{ig^{(N)}}{\hbar} H_\mu, \quad (139-1)$$

$$L_{\mu\nu} = \nabla_\nu H_\mu - \nabla_\mu H_\nu + \frac{ig^{(N)}}{\hbar} [H_\nu, H_\mu],$$

$$Z_{\mu\nu\rho} = S_{\rho\mu\nu} = \nabla_\nu L_{\rho\mu} - \nabla_\mu L_{\rho\nu} + \frac{ig^{(N)}}{\hbar} [H_\nu L_{\rho\mu} - H_\mu L_{\rho\nu}]. \quad (139-2)$$

where $g^{(N)}$ is the coupling constant.

Corollary 5. According to the unique structures of equations (135-1) – (135-4), we can conclude that magnetic monopoles do not exist in nature.

Remark 6. The Generalized form of Einstein field equations could be obtained from equations (112-6), (113-6), (135-6). Using the (second) Bianchi identity and the conventions (107) we get

$$\nabla_\mu R^\mu_{\nu\rho\sigma} = \nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho} \quad (140)$$

Now from (140) and equations (112-6), (113-6), (135-6) and the assumption

$$J_{\nu\rho\sigma} = -8\pi(\nabla_\sigma T_{\nu\rho} - \nabla_\rho T_{\nu\sigma}) + 8\pi B(\nabla_\sigma T g_{\nu\rho} - \nabla_\rho T g_{\nu\sigma}) \quad (141)$$

where $T_{\mu\nu}$ is the stress-energy tensor ($T = T^\mu_\mu$), and $g_{\mu\nu}$ is the metric tensor and B is a constant, we get

$$R_{\mu\nu} = -8\pi(T_{\mu\nu} - B T g_{\mu\nu}) - \frac{im_0}{\hbar} K_{\mu\nu} - q g_{\mu\nu} \quad (142)$$

where q is a constant (that appears naturally, when we obtain equation (142)), and $K_{\mu\nu} = \nabla_\mu k_\nu$ (where k_ν has been defined in (115-1)), and $K_{\mu\nu} = -K_{\nu\mu}$, $K^\mu_\mu = 0$ (due to its complex coefficient); Equation (142) for (112-6) (two dimensional case), specially, takes the following form

$$8\pi T_{\mu\nu} + \frac{im_0}{\hbar} K_{\mu\nu} = \frac{1}{2}(8\pi T)g_{\mu\nu}, \quad (143-1)$$

$$R_{\mu\nu} = -\frac{1}{2}(8\pi T)g_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu}; \quad (143-2)$$

$$R - \Lambda = -8\pi T \quad (144)$$

where $\Lambda = 2q$, $B = 0$ and Λ is the cosmological constant.

Regarding equation (113-6) (three dimensional case), the specific form of the field equation (142) is as follow

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi T_{\mu\nu} - \frac{im_0}{\hbar} K_{\mu\nu} - \Lambda g_{\mu\nu} \quad (145)$$

where $\Lambda = \frac{1}{2}q$, $B = 1$ and Λ is the cosmological constant.

And concerning equation (135-6) (four dimensional case), the field equation (142) takes the following specific form, as well

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi T_{\mu\nu} - \frac{im_0}{\hbar} K_{\mu\nu} - \Lambda g_{\mu\nu} \quad (146)$$

where $\Lambda = q$, $B = \frac{1}{2}$ and Λ is the cosmological constant.

In equations (142) – (146) we may have the (total) conservation law as follows¹

$$\nabla_{\mu}(T^{\mu\nu} + \frac{im_0}{\hbar} Z^{\mu\nu}) = 0 \quad (147)$$

1. Meanwhile, according to the second Bianchi identity, from the equations (113-5) and (135-5) we may have

$a, b \neq 0$: $\frac{im_0}{\hbar} R_{ab\rho\sigma} = 0$; that in the case $m_0 \neq 0$ (as the invariant mass of a supposed gravitational force carrier particle), there may

appear some additional conditions for the components of the Riemann tensor.

Remark 7. In the special relativity conditions, equations (112-1) – (112-4), (113-1) – (113-4) and (135-1) – (135-4) also could be written as follows

$$(i\hbar\alpha^\mu\partial_\mu - Im_0)[F] = 0, \quad (148)$$

$$(i\hbar\alpha^\mu\partial_\mu - Im_0)[Z] = 0. \quad (149)$$

where I is the identity matrix; and where for equations (112-1) – (112-4) (i.e. two dimensional case) we have

$$\begin{aligned} \alpha^0 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \alpha^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ [F] &= \begin{bmatrix} F_{10} \\ \varphi^{(E)} \end{bmatrix}, \quad [Z] = \begin{bmatrix} Z_{10\rho} \\ \varphi_\rho^{(N)} \end{bmatrix}. \end{aligned} \quad (150)$$

and for equations (113-1) – (113-4) (i.e. three dimensional case) we have

$$\begin{aligned} \alpha^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ [F] &= \begin{bmatrix} F_{10} \\ F_{02} \\ F_{21} \\ \varphi^{(E)} \end{bmatrix}, \quad [Z] = \begin{bmatrix} Z_{10\rho} \\ Z_{02\rho} \\ Z_{21\rho} \\ \varphi_\rho^{(N)} \end{bmatrix}. \end{aligned} \quad (151)$$

and for equations (135-1) – (135-4) (i.e. four dimensional case) we have

$$\begin{aligned}
\alpha^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\alpha^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$[F] = \begin{bmatrix} F_{10} \\ F_{20} \\ F_{30} \\ 0 \\ F_{23} \\ F_{31} \\ F_{12} \\ \varphi^{(E)} \end{bmatrix}, \quad [F] = \begin{bmatrix} Z_{10\rho} \\ Z_{20\rho} \\ Z_{30\rho} \\ 0 \\ Z_{23\rho} \\ Z_{31\rho} \\ Z_{12\rho} \\ \varphi_\rho^{(N)} \end{bmatrix}. \quad (152)$$

All these matrices (including previous matrices) correspond to Clifford algebras [10].

Corollary 5. In some special cases such as: $(m_0 \neq 0)$ and $(\varphi^{(E)} = 0, \varphi^{(N)} = 0)$, the equations (148) and (149) (as well as the equations (112-1) – (113-6) and (135-1) – (135-6)) are turned into some (wave) equations similar to Pauli and Dirac equations. Which we can suppose that they are the corrected and generalized forms of Pauli equation (in two dimensional case), and Dirac equation (in three dimensional case) and so on.

4. Conclusion

In the section 2., since the set of algebraic axioms (17) – (23) for integers, have been formulated in terms of the quadratic $(n \times n)$ matrices (with arbitrary “ n ”), we may conclude that for complete representing the algebraic property of integers, necessarily and sufficiently, the quadratic matrices $(n \times n)$: $([a_{ij}]_{n \times n}, [b_{ij}]_{n \times n}, [c_{ij}]_{n \times n}, \dots \in Z_{n \times n})$ should be applied. The (old) algebraic axioms (10) – (16-2) that had been formulated in terms of the elements: $a_1, a_2, a_3, \dots \in Z$, (that in fact, we can assume that these elements are 1×1 matrices such as: $[a]_{1 \times 1} (\equiv a), [b]_{1 \times 1} (\equiv b), [c]_{1 \times 1} (\equiv c), \dots \in Z_{1 \times 1} (\equiv Z)$) are not sufficient for complete introducing the algebraic properties of integers.

In the section 3. of this article we derive the general and unique forms of all field equations governing the fundamental forces of nature, including equations (112-1) – (113-6) and (135-1) – (135-6). This derivation was based on a new mathematical approach (represented in the section 2., concerning the algebraic properties of the domain of integers), as well as the assumption of discreteness of the components of the relativistic n -momentum. The obtained laws include three separate types of the field tensor equations (and only three), representing the force of gravity, the electromagnetic force, and the (strong) nuclear force (for dimensions $D \geq 2$). In some special conditions, these equations could be turned into the (wave) equations that are similar to Pauli and Dirac equations. In actual fact, they are the corrected and generalized forms of the current field equations including Maxwell equations, Yang-Mills equations and Einstein equations, as well as (in some special conditions) Pauli equation, Dirac equation, and so on. In addition, according to obtained laws, we also showed that magnetic monopoles could not exist in nature. The results obtained in the section 3. demonstrate the efficiency of the theory of linearization (and its algebra) and a wide range of its possible applications. The proposed mathematical structure in the section 2., doubtless can be useful in many fields where the discreteness of the relevant quantities are supposed and applied.

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